DEVOIR 1. HOLOMORPHIC MAPS AND COMPLEX MANIFOLDS

Exercises with ★: hand in only these exercises.
Exercises with ★★: don't hand in, try to read some references and convince yourself.

Exercise 1. Let f be a meromorphic function on an open and connected subset U in $\mathbb{C}$.

(a) Show that for any compact set $K \subset U$, the set of zeros and poles of f in K are finite.

(b) For $x \in U$, show that there exists an integer $k_x \in \mathbb{Z}$ such that, in a neighborhood U_x of x,
$$f(z) = (z - x)^{k_x} h(z)$$
with $h : U_x \to \mathbb{C}$ holomorphic and invertible.

(c) Let $\mathbb{D} \subset U$ be a disk with center $x \in U$, such that x is the only pole and zero of f in $\mathbb{D}$. Show that
$$k_x = \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{df}{f}.$$

(d) Show that if $(a_1, \cdots, a_d) \in \mathbb{D}^d$, are the zeros (counted with multiplicities) of a holomorphic function $f : \mathbb{D} \to \mathbb{C}$, and $m \in \{0, \cdots, d\}$, then
$$\sum_{i=1}^{d} a_i^m = \frac{1}{2\pi i} \int_{\partial \mathbb{D}} z^m \frac{f'(z)}{f(z)} dz.$$

Exercise 2. Let f be a holomorphic function on the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$. Suppose that $|f| \leq M$ and $D^\alpha f(0) = 0$ for all $|\alpha| \leq k$. Show that $|f(z)| \leq M\|z\|^k$ for all $z \in \mathbb{B}^n$.
Hint: you can consider the function $g_z(t) = f\left(\frac{tz}{\|z\|}\right)$.

Exercise 3. ★ Let U be an open connected subset of $\mathbb{C}^2$. Consider $f : U \to \mathbb{C}$ a holomorphic function and $Z(f) := \{x \in U \mid f(x) = 0\}$.

(a) Show that if f doesn't vanish on $U \setminus \{z_1 = z_2 = 0\}$, then $Z(f)$ is empty.

(b) Show that $U \setminus Z(f)$ is either connected and dense or empty.
Hint: if $U \setminus Z(f) = U_1 \cup U_2$ *is not connected, apply Riemann's extension theorem to the function* $g \in \mathcal{O}(U \setminus Z(f))$ *such that* $g = 1$ *on* U_1 *and* $g = 0$ *on* U_2.

Exercise 4. Let U be an open subset of $\mathbb{C}^n$, which we identify with $\mathbb{R}^{2n}$ via the map
$$\mathbb{C} \ni z_j = x_j + iy_j \mapsto (x_j, y_j) \in \mathbb{R}^2,$$
where $z_j = x_j + iy_j$ is the standard coordinates on U for $1 \leq j \leq n$. Consider a smooth map $f : U \to \mathbb{C}^m = \mathbb{R}^{2m}$, also interpreted as a map to $\mathbb{R}^{2m}$. The *complex Jacobian* of f is defined as the matrix
$$J_{\mathbb{C}}(f) = \left(\frac{\partial f_i}{\partial z_j}\right)_{1 \leq i \leq m, 1 \leq j \leq n}.$$

(a) With respect to the standard coordinates $w_j = u_j + iv_j$ on $\mathbb{C}^m$, compute the matrix of
$$df : T\mathbb{R}^{2n} \to T\mathbb{R}^{2m}.$$

(b) Find the matrix of the induced map
$$df_{\mathbb{C}} : T\mathbb{R}^{2n} \otimes \mathbb{C} \to T\mathbb{R}^{2m} \otimes \mathbb{C}$$
in terms of the frames $\left\{\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_j}\right\}$ and $\left\{\frac{\partial}{\partial w_k}, \frac{\partial}{\partial \bar{w}_k}\right\}$.

(c) Is there a relation between $J_{\mathbb{C}}(f)$ and $df_{\mathbb{C}}$? What happens when f is holomorphic?

(d) Show that when f is holomorphic and $n = m$, we have
$$\det df = |\det J_{\mathbb{C}}(f)|^2.$$

(e) Use this to prove that a complex structure induces a natural orientation on the underlying differentiable manifold.

(f) Give an example of an even-dimensional manifold that does not admit a complex structure. Could it admit an almost complex structure?

Exercise 5. Suppose that X is a complex manifold and Γ is a discrete group acting freely and properly discontinuously by biholomorphisms on X.

(a) Show that X/Γ admits the structure of a complex manifold such that the map $\pi : X \to X/\Gamma$ is holomorphic and locally biholomorphic.
(b) Deduce that the Hopf surface defined as the quotient $\mathbb{C}^2 \setminus \{0\}/\sim$, where $(w_1, w_2) \sim (z_1, z_2)$ if $w_j = 2^s z_j$ for some fixed $s \in \mathbb{Z}$, is a complex manifold.
(c) Show that the Hopf surface is diffeomorphic to $\mathbb{S}^3 \times \mathbb{S}^1$.

Exercise 6. ★ The n-dimensional complex projective space $\mathbb{P}^n$ is the quotient

$$(\mathbb{C}^{n+1} \setminus \{0\})/\sim$$

where $z \sim w$ if $\exists \lambda \in \mathbb{C}^*$ such that $\lambda z = w$.

(a) Show that $\mathbb{P}^n$ is a complex manifold of dimension n.
(b) Show that $\mathbb{P}^n$ is compact and connected.
(c) Give a diffeomorphism between $\mathbb{P}^1$ and $\mathbb{S}^2$.
(d) (optional) Convince yourself that for any $x \in \mathbb{P}^n$, $T_x\mathbb{P}^n \simeq \mathrm{Hom}_{\mathbb{C}}(x, \mathbb{C}^{n+1}/x)$ as a vector space.

Exercise 7. ★ Let $f : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function and $0 \in \mathbb{C}$ is a regular point (i.e. the complex Jacobian $J_{\mathbb{C}}(f)$ is surjective at any point in $f^{-1}(0)$). Show that $f^{-1}(0)$ is a complex submanifold in $\mathbb{C}^n$.

Exercise 8. ★ Let F be a homogeneous polynomial in $(n+1)$ variables. Show that the set

$$Z = \{[z_0 : \cdots : z_n] \in \mathbb{P}^n \mid F(z_0, \cdots, z_n) = 0\}$$

is well-defined. Considering the map $f : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}$ induced by F, show that if 0 is a regular value of f then Z is a complex submanifold of $\mathbb{P}^n$.

Exercise 9. ★ Let X be a compact complex manifold. Show that if a holomorphic function $f : X \to \mathbb{C}$ vanishes on an open subset of X, then it must be identically zero.

Exercise 10. Let X be an n-dimensional complex manifold and $Y \subset X$ a real submanifold of dimension $2k$. Show that Y is a complex submanifold if and only if, locally, it coincides with the vanishing locus of $n-k$ independent holomorphic functions.

Exercise 11 (Blow-ups, reading exercise: Voisin's book Section 3.3.3)**.** ★★ Let X be an n-dimensional complex manifold and $Y \subset X$, a closed complex submanifold of dimension $n-k < n$. Let $U \subset X$ be an open set such that $U \cap Y = Z(f_1^U) \cap \cdots \cap Z(f_k^U)$ where $f^U = (f_1^U, \cdots, f_k^U) : U \to \mathbb{C}^k$ is holomorphic (and 0 is a regular value).

(a) Show that if $g^U = (g_1^U, \cdots, g_k^U)$ is another system of holomorphic functions defining $Y \cap U$ then there exists a matrix valued function $M_{fg}^U : U \to \mathrm{GL}(k, \mathbb{C})$ with holomorphic components and such that $g^U = M_{fg}^U f^U$ pointwise. Moreover, M_{fg}^U is uniquely determined by f^U and g^U along Y.
(b) Check that $\widetilde{U}_Y := \{([w_1, \cdots, w_k], z) \in \mathbb{P}^{k-1} \times U \mid w_j f_i^U(z) = w_i f_j^U(z), i, j = 1, \cdots, k\}$ is complex submanifold of $\mathbb{P}^{k-1} \times U$ and that the projection over the second factor $\tau_U : \widetilde{U}_Y \times U$ is an isomorphism over $U \setminus Y$. What is $\tau_U^{-1}(y)$ for $y \in U \cap Y$?
(c) Use (a) to show that if $V \subset X$ is another subset defining $\widetilde{V}_Y$ as above we get a natural biholomorphism

$$\phi_{UV} : \tau_U^{-1}(U \cap V) \to \tau_V^{-1}(U \cap V).$$

In particular, using these ϕ_{UV} to glue those open complex charts, we get a complex manifold $\widetilde{X}_Y$ called the blow-up X along Y. Show that the maps $\tau_U : \widetilde{U}_Y \to U$ glued together give a surjective map $\tau : \widetilde{X}_Y \to X$ which is a biholomorphism over $\widetilde{X}_Y \setminus \tau^{-1}(Y)$.
(d) Show that for $X = \mathbb{C}^{n+1}$ and $Y = \{0\}$ there is a surjective map $\pi : \widetilde{X}_Y \to \mathbb{P}^n$.