DEVOIR 2. COMPLEX MANIFOLDS AND DIFFERENTIAL CALCULUS

Exercises with ★: hand in only these exercises (Exercises 3, 7, 10, 14, 16).
Exercises with ★★: not for return, but try to do it by yourself.
Exercises without ★ : these are standard exercises; if you don't know them, it's important to learn them.

1. Review of differential manifolds and differential calculus

Exercise 1. Consider the 2-torus $\mathbb{T}^2$ and the local diffeomorphism $\Phi : \mathbb{R}^2 \to \mathbb{T}^2$ defined by $\Phi(\theta_1, \theta_2) = (e^{2\pi i\theta_1}, e^{2\pi i\theta_2})$.

(a) Exhibit a condition on a vector field $X \in \Gamma(T\mathbb{R}^2)$ for $\Phi_* X$ to be a vector field on $\mathbb{T}^2$.
(b) Deduce $T\mathbb{T}^2 \simeq \mathbb{T}^2 \times \mathbb{R}^2$.
(c) More generally, show that the tangent space of an n-dimensional manifold is trivial if and only if there exist n nowhere vanishing vector fields linearly independent at each point.

Exercise 2. Define the n-sphere $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} \mid \sum_{i=1}^n x_i^2 = 1\}$.

(a) Use the implicit function theorem to prove that $\mathbb{S}^n$ is a smooth manifold of dimension n.
(b) Show that $T\mathbb{S}^n \simeq \{(x, y) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid |x| = 1, \langle x, y\rangle = 0\}$.
(c) Exhibit an explicit diffeomorphism between $T\mathbb{S}^{n-1}$ and $\{z \in \mathbb{C}^n \mid \sum_{j=1}^n z_j^2 = 1\}$.
(d) We denote by $X \wedge Y$ the cross product of $\mathbb{R}^3$. Show that for $X_p, Y_p \in T_p\mathbb{S}^2$, $\omega_p(X_p, Y_p) = \langle p, X_p \wedge Y_p\rangle$ defines a non-degenerate closed 2-form on $\mathbb{S}^2$.

Exercise 3 (*Fibre bundle construction*)**.** ★ Let M be a smooth manifold with a given open cover $M = \bigcup_{i\in I} U_i$ and let $\psi_{ij} : U_i \cap U_j \to \mathrm{GL}(k, \mathbb{R})$ be smooth maps satisfying the *cocycle condition*

$$\psi_{ik}(x) = \psi_{ij}(x) \cdot \psi_{jk}(x) \qquad \forall x \in U_i \cap U_j \cap U_k, \tag{cc}$$

(in particular, $\psi_{ii}(x) = \mathrm{Id}$ and $\psi_{ji}(x) = \psi_{ij}(x)^{-1}$).

(a) Use these maps ψ_{ij} to construct a vector bundle E of rank k over M with transition maps. *Hint: consider the set* $\bigsqcup_i \{(i, x, v) \mid i \in I, x \in U_i, v \in \mathbb{R}^k\}$ *and quotient by a suitable relation.*
(b) Show that given another set of transition maps $\widetilde{\psi}_{ij} : U_i \cap U_j \to \mathrm{GL}(k, \mathbb{R})$ satisfying (cc), the vector bundle obtained $\widetilde{E}$ is isomorphic (as bundle) to E if and only if there exists maps $h_i : U_i \to \mathrm{GL}(k, \mathbb{R})$ such that $\widetilde{\psi}_{ij}(x) = h_i(x)^{-1} \cdot \psi_{ij}(x) \cdot h_j(x)$ for all $x \in U_i \cap U_j$.
(c) Exhibit the trivialization and transition maps of the dual vector bundle E^* in terms of those of E.

Exercise 4. Let M be a manifold and $X, Y, Z \in \Gamma(TM)$ be three vector fields.

(a) Show that if $[X, W]$ for any $W \in \Gamma(TM)$ then $X \equiv 0$.
(b) Show that $\phi_*[X, Y] = [\phi_* X, \phi_* Y]$ for any $\phi \in \mathrm{Diffeo}(M)$.
(c) Deduce the Jacobi identity $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.
(d) Denote by $\phi_t^X, \phi_t^Y \in \mathrm{Diffeo}(M)$ the flow of X and Y respectively. Show that ϕ_t^X and ϕ_s^Y commute for all t, s small enough if and only if $[X, Y] = 0$.

Exercise 5. Let M be a manifold and $X \in \Gamma(TM)$ be a vector field. We will prove the Cartan formula:

$$\mathcal{L}_X = d \circ \iota_X + \iota_X \circ d.$$

(a) Show that it is sufficient to prove the Cartan formula on 0-forms (thanks to Leibniz).
(b) Prove the Cartan formula on 0-forms.
(c) Let $\alpha \in \Gamma(T^*M)$ be a 1-form and $X, Y \in \Gamma(TM)$. Show that $d\alpha(X, Y) = X.\alpha(Y) - Y.\alpha(X) - \alpha([X, Y])$.

Exercise 6 (*Frobenius Theorem*)**.** ★★ Let $U \subset \mathbb{R}^n$ be an open set. We will show the following: Let $D \subset TU$ be a subbundle of non-zero rank $k < n$. For any $p \in U$, there exists a submanifold $N \subset U$ with $p \in N$ and $T_pN = D_p$ if and only if

$$\forall X, Y \in \Gamma(D), \quad [X, Y] \in \Gamma(D). \tag{$*$}$$

(a) Show that $(*)$ is necessary.
(b) Let $X_1, \cdots, X_k \subset \Gamma(D)$ be a local frame. Use $(*)$ to produce a local frame $Y_1, \cdots, Y_k \in \Gamma(D)$ such that $[Y_i, Y_j] \equiv 0$ for all i, j.

(c) Conclude.

2. Complex setting

Exercise 7. ★ Let (M, J) be an almost complex manifold.

(a) Show that the Nijenhuis tensor $N_J(X,Y) = \frac{1}{4}([JX, JY] - J[JX, Y] - J[X, JY] - [X, Y])$ is a tensor (i.e. is $\mathcal{C}^\infty(M)$-linear).
(b) Show that $T^{1,0}M$ is closed under the Lie bracket if and only if N_J vanishes identically.
(c) Deduce that any almost complex structure on a manifold M of (real) dimension 2, $T^{1,0}M$ is closed under the Lie bracket.

Exercise 8. Let (M, J) be a complex manifold (i.e. J is integrable) show that $T^{1,0}M$ is (naturally) a holomorphic vector bundle over M.

Exercise 9. Let (M, J) be a complex manifold.

(a) Show that $\overline{\partial\alpha} = \bar{\partial}\overline{\alpha}$.
(b) Deduce that a real (p,p)-form $\alpha \in \mathcal{A}^{p,p}(M) \cap \mathcal{A}^{2p}(M)$ is $\bar{\partial}$-closed (resp. exact) and if and only if it is ∂-closed.
(c) Formulate the ∂ and $\bar{\partial}$-Poincaré Lemmas

Exercise 10. ★ Let $f : M \to N$ be a holomorphic map between complex manifolds. Prove that if α is a (p,q)-form on N then $f^*\alpha$ is a (p,q)-form on M. Given an example where this fails if f is not holomorphic. Using this to show that f induces a homomorphism

$$f^* : H^{p,q}_{\bar{\partial}}(N) \to H^{p,q}_{\bar{\partial}}(M)$$

given by $f^*[\alpha] = [f^*\alpha]$ for $\alpha \in \mathcal{A}^{p,q}(N)$ with $\bar{\partial}\alpha = 0$.

Exercise 11. ★★ Consider the natural map $\pi : \mathbb{C}^{n+1}\setminus\{0\} \to \mathbb{P}^n$ defining $\mathbb{P}^n$. Assume that $\alpha \in \mathcal{A}^{p,0}(\mathbb{P}^n)$ is $\bar{\partial}$-closed (i.e. α is a holomorphic p-form).

(a) Show that $\pi^*\alpha$ extends as a $\bar{\partial}$-closed $\beta \in \mathcal{A}^{p,0}(\mathbb{C}^{n+1})$.
(b) Show that β is homogeneous (i.e. for $\lambda \in \mathbb{C}^*$, denote the dilation of $\mathbb{C}^{n+1}$ by $\gamma_\lambda(z) = \lambda z$, then $\gamma_\lambda^*\beta = \beta$).
(c) Writing $\beta = \sum_I f_I(z)dz_I$, show that it implies that $f \equiv 0$ on $\mathbb{C}^{n+1}$.
(d) Conclude that $H^{p,0}_{\bar{\partial}}(\mathbb{P}^n) = 0$ if $p > 0$.

Exercise 12. Let M be a simply connected compact complex manifold. Prove that $H^{1,0}(M) = 0$.
Hint: given a holomorphic 1-form α, integrate it along paths with a fixed starting point to define a holomorphic map $f : X \to \mathbb{C}$ with $df = \alpha$.

Exercise 13. Let (M, J) be a complex manifold. For any J-invariant real 2-form $\psi \in \mathcal{A}^{1,1}(M) \cap \mathcal{A}^2(M)$, check that $b_\psi \in \Gamma((T^*M)^{\otimes 2})$, defined as $b_\psi(X,Y) = \psi(X, JY)$ is bilinear, J-invariant and symmetric. We say that ψ is positive if b_ψ is positive definite at each point.

(a) On $\mathbb{C}^n$, show that $\omega := \frac{i}{2}\sum_{j=1}^n dz_j \wedge d\bar{z}_j$ is positive; in particular that b_ω is the standard metric on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$.
(b) Show that $\omega = \frac{1}{2}\mathrm{i}\partial\bar{\partial}r^2$ for $r^2 = \sum_{j=1}^n |z_j|^2$
(c) More generally, for an open set $U \subset \mathbb{C}^n$, check that if $f \in \mathcal{C}^2(U, \mathbb{R})$ is a strictly convex function, $\mathrm{i}\partial\bar{\partial}f$ is positive on U.

Exercise 14. ★ Let $E \to M$ be a rank k complex vector bundle over M whose transition functions with respect to some open cover $(U_\alpha)_\alpha$ of M are $(g_{\alpha\beta})_{\alpha,\beta}$. Show that a section $\sigma : M \to E$ of E can be identified with a collection $(\sigma_\alpha)_\alpha$ of smooth map $\sigma_\alpha : U_\alpha \to \mathbb{C}^k$ satisfying $\sigma_\alpha = g_{\alpha\beta}\sigma_\beta$ on $U_\alpha \cap U_\beta$.

Exercise 15 (*The tautological bundle and the hyperplane bundle*)**.** ★★ Let L be the complex line bundle $\pi : L \to \mathbb{P}^n$ whose fibre L_x over some point $x \in \mathbb{P}^n$ is the complex line x in $\mathbb{C}^{n+1}$. Let L^* be the dual line bundle to L.

(i) Prove that L is a holomorphic line bundle (*hint: use the local transitions*) and show that L has no non-trivial holomorphic section.
(ii) For any given $\alpha \in \mathbb{C}^{n+1} \setminus \{0\}$, show that by the restriction to L_x that α defines a section s_α to L^*. Conclude that the space of global holomorphic sections of L^* has at least dimension $n+1$. What is the zero locus of s_α in $\mathbb{P}^n$? Given $k \geq 0$, interpret any homogeneous polynomial of order k on $\mathbb{C}^{k+1}$ as a section of $(L^*)^{\otimes k}$.

Exercise 16. ★ Let $\pi : E \to M$ be a rank r holomorphic vector bundle. For a local frame of holomorphic sections $s_1, \cdots, s_k$ on $U \subset M$, we define $\bar{\partial}_E : \Gamma(\Lambda^{p,q}U \otimes E) \to \Gamma(\Lambda^{p,q+1}U \otimes E)$

$$\bar{\partial}_E \left(\sum_{j=1}^{k} \alpha_j \otimes s_j \right) = \sum_{j=1}^{k} \bar{\partial}\alpha_j \otimes s_j$$

(a) Show that $\bar{\partial}_E$ does not depend on the chosen frame and extends as a well-defined operator $\bar{\partial}_E : \Gamma(\Lambda^{p,q}M \otimes E) \to \Gamma(\Lambda^{p,q+1}M \otimes E)$
(b) Show that $\bar{\partial}_E^2 = 0$.
(c) Which group of cohomology is associated with that complex coincides with the space of globally defined holomorphic sections?