DEVOIR 3. KÄHLER METRICS AND HOLOMORPHIC VECTOR BUNDLES

Exercises with ★: hand in only these exercises (Exercises 4, 7, 8, 9, 10).
Exercises with ★★: not for return, but give it a try or read some references and convince yourself.
Exercises without ★: these are standard exercises; if you don't know them, it's important to learn them.

Exercise 1. Let (M,ω) be a Kähler manifold and f a C^2 compactly supported real-valued function on M. Show that for $\varepsilon \in \mathbb{R}$ small enough, $\omega + \varepsilon \mathrm{i}\partial\bar\partial f$ is a Kähler metric on M.

Exercise 2. Let $U \subset \mathbb{C}^n$ be an open subset and $\omega = \frac{\mathrm{i}}{2}\sum_{j,k=1}^n h_{j\bar k}\mathrm{d}z_j \wedge \mathrm{d}\bar z_k \in \mathcal{A}^{1,1}(U)$. Show that $\omega^n = (\frac{\mathrm{i}}{2})^n \det(h_{j\bar k})\mathrm{d}z_1 \wedge \mathrm{d}\bar z_1 \wedge \cdots \wedge \mathrm{d}z_n \wedge \mathrm{d}\bar z_n$.

Exercise 3 (Local $\partial\bar\partial$-lemma)**.** Let $\Delta \subset \mathbb{C}^n$ be a polydisk. Given a $(1,1)$-form $\alpha \in \mathcal{A}^{1,1}(\Delta) \cap \mathcal{A}^2_{\mathbb{R}}(\Delta)$, $\mathrm{d}\alpha = 0$, show that there exists $f \in \mathcal{C}^\infty(\Delta)$ such that $\alpha = \mathrm{i}\partial\bar\partial f$.

Exercise 4. ★ Let (M, J, ω) be a compact Kähler manifold. Check that the following two Kähler identities are equivalent:

$$[\Lambda_\omega, \bar\partial] = -\mathrm{i}\partial^* \qquad\qquad [\bar\partial^*, L_\omega] = \mathrm{i}\partial.$$

Exercise 5. On $\mathbb{C}^n$ with its standard Kähler metric and for $f \in C^2(\mathbb{C}^n, \mathbb{R})$, check that $\Delta_{\mathrm{d}} f$ is (a multiple of) the Laplacian that you learned in complex analysis or PDE courses.

Exercise 6. Let (M, J, ω) be a compact Kähler manifold. Show that Δ_{d}, Δ_∂, $\Delta_{\bar\partial}$ commute with $*$, L_ω, Λ_ω, ∂, ∂^*, $\bar\partial$, and $\bar\partial^*$.

Exercise 7. ★ Let M be a comapct Kähler manifold and let $\pi : L \to M$ be a holomorphic line bundle with a hermitian metric h, Chern connection ∇_h and curvature $F_{\nabla_h} \in \mathcal{A}^{1,1}(M) \cap \mathcal{A}^2_{\mathbb{R}}(M)$.

1. Given a function $f \in C^2(M, \mathbb{R})$, compute the curvature of the Chern connection associated to the hermitian metric $e^{-f}h$ in terms of F_{∇_h} and f.
2. Deduce that the class F_{∇_h} in $H^{1,1}_{\bar\partial}(M)$ does not depend on h.
3. Using the $\partial\bar\partial$-lemma, show that every form in the class $[F_{\nabla_h}]$ is the curvature of a hermitian metric on L.

Exercise 8. ★ Let $L \to M$ be a holomorphic line bundle and L^{-1} its dual bundle. Given a connection ∇ on L we define a operator $\nabla^* : \Gamma(L^{-1}) \to \Gamma(T^*M \otimes L^{-1})$ by setting $(\nabla^*_X\sigma)(s) = X(\sigma(s)) - \sigma(\nabla_X s)$ for all $s \in \Gamma(L)$ and $X \in \Gamma(TM)$. Show that ∇^* is a connection on L^{-1} and that $F_{\nabla^*} = -F_\nabla$.

Exercise 9. ★ Given a compact complex manifold M and the set of isomorphic classes of holomorphic line bundles together with the tensor product defines an abelian group $\mathrm{Pic}(M)$. Show that the map $c_1 : \mathrm{Pic}(M) \to H^{1,1}_{\bar\partial}(M)$ given by $c_1(L) = [\frac{\mathrm{i}}{2\pi}F_{\nabla_h}]$ for any hermitian metric h on L, is a well-defined group homomorphism. In particular, $c_1(L^{\otimes k}) = kc_1(L)$.

Exercise 10 (The tautological line bundle, again)**.** ★ Let $L = \mathcal{O}_{\mathbb{P}^n}(-1)$ be the tautological line bundle $\pi : L \to \mathbb{P}^n$ whose fibers L_x over the point $x \in \mathbb{P}^n$ is the complex line $x \in \mathbb{C}^{n+1}$.

1. Check that L is naturally a holomorphic subbundle of the trivial bundle $\mathbb{P}^n \times \mathbb{C}^{n+1}$ and thus that the standard hermitian metric on $\mathbb{C}^{n+1}$ provides a hermitian metric on L.
2. Compute the associated Chern curvature on L and show that it gives $-\mathrm{i}\lambda\omega_{\mathrm{FS}}$ for some negative constant $\lambda < 0$ (here ω_{FS} is the Fubini-Study metric on $\mathbb{P}^n$).
   (*Some guide: You can look in Voisin's book for example.*)
3. Deduce the curvature of the Chern connection induced on the dual $L^{-1} = \mathcal{O}_{\mathbb{P}^n}(1)$ is a positive multiple of the Fubini-Study metric (cf. Exercise 8).

Exercise 11. Let M be the blow-up of $\mathbb{C}^{n+1}$ at the origin (see Devoir 1). Give a biholomorphism between M and the total space of the tautological line bundle over $\mathbb{P}^n$.

Exercise 12. Let E_1 and E_2 be two vector bundles on M endowed with connections ∇^{E_1} and ∇^{E_2} respectively. Let s_1 and s_2 be local sections of E_1 and E_2, respectively. Define

$$\nabla^{E_1\oplus E_2}(s_1 \oplus s_2) = \nabla^{E_1}(s_1) \oplus \nabla^{E_2}(s_2), \quad \text{and} \quad \nabla^{E_1\otimes E_2}(s_1 \otimes s_2) = \nabla^{E_1}(s_1) \otimes s_2 + s_1 \otimes \nabla^{E_2}(s_2).$$

1. Check that $\nabla^{E_1\oplus E_2}$ and $\nabla^{E_1\otimes E_2}$ are connections on $E_1 \oplus E_2$ and $E_1 \otimes E_2$ respectively.

(2) Show that $F_{\nabla^{E_1\oplus E_2}} = F_{\nabla^{E_1}} \oplus F_{\nabla^{E_2}}$, and $F_{\nabla^{E_1\otimes E_2}} = F_{\nabla^{E_1}} \otimes \mathrm{Id}_{E_2} + \mathrm{Id}_{E_1} \otimes F_{\nabla^{E_2}}$.
(3) Note that $\mathrm{End}(E) = E \otimes E^*$. Deduce that $F_{\nabla^{\mathrm{End}(E)}} = [F_{\nabla^E}, \bullet]$.

Exercise 13 (You can follow Huybrechts's book section 4.A)**.** ★★ Let (M, J, ω) be a Kähler manifold. Show that the ∇^{C} the Chern connection on $T^{1,0}M$ correspond to ∇^{LC} the Levi-Civita connection on TM under the isomorphism $TM \simeq T^{1,0}M$ induced by $X \mapsto \frac{1}{2}(X - iJX)$.

Exercise 14 (Projective bundles (you can follow Voisin's book section 3.3.2))**.** ★★ Let $\pi : E \to M$ be a rank $r+1$ holomorphic bundle over a compact complex manifold M and denote $E^\times$ the bundle over M obtained by removing the zero section of E, i.e. $E^\times = E \setminus \{0\}$-section. We define $\mathbb{P}(E)$ to be the quotient of $E^\times$ by the natural fiberwise $\mathbb{C}^*$-action.

(1) Check that the map π descends as well-defined map $\hat{\pi} : \mathbb{P}(E) \to M$ so that $\hat{\pi}^{-1}(p)$ coincides with the set of complex lines in the fiber $E_p = \pi^{-1}(p)$ of E. In particular, $\hat{\pi}^{-1}(p) \simeq \mathbb{P}^r$.
(2) Check that a holomorphic atlas $\{U_i\}_{i\in I}$ of M trivialising E provides identifications $\hat{\pi}^{-1}(U_i) \simeq U_i \times \mathbb{P}^r$ and that the projective morphism associated to the transition maps of E give transition maps of the atlas $\hat{\pi}^{-1}(U_i)$ on $\mathbb{P}(E)$.
(3) Deduce that this provides a complex structure on $\mathbb{P}(E)$ such that $\hat{\pi} : \mathbb{P}(E) \to M$ is holomorphic.
(4) Consider a holomorphic line bundle $p : \hat{\pi}^*E \to \mathbb{P}(E)$. Remember that the fibers of $\hat{\pi}^*E$ over $x \in \mathbb{P}(E)$ are $E_{\hat{\pi}(x)}$. Convince yourself that $\mathcal{O}_{\mathbb{P}(E)}(-1) := \{\xi \in \hat{\pi}^*E \mid \xi \in p(\xi)\}$ is a holomorphic line bundle over $\mathbb{P}(E)$ and show that for $p \in M$ the pull-back of $\mathcal{O}_{\mathbb{P}(E)}(-1)$ on the fiber $\mathbb{P}(E)_p = \hat{\pi}^{-1}(p) \simeq \mathbb{P}^r$ coincides with the tautological line bundle of $\mathbb{P}^r$.
(5) Consider now a hermitian metric h on E. Show that it provides a hermitian metric, still denoted h, on $\mathcal{O}_{\mathbb{P}(E)}(-1)$.
(6) Show that the restriction of $-F_{\nabla_h}$ to a fiber $\mathbb{P}(E)_p$ is positive for any $p \in M$.
(7) Conclude that if (M, ω) is compact and Kähler there exists $\lambda > 0$ such that $\lambda\pi^*\omega - F_{\nabla_h}$ is Kähler on $\mathbb{P}(E)$.

Exercise 15 (Blow-ups, again (you can follow Voisin's book section 3.3.3))**.** ★★ Let X be an n-dimensional complex manifold and $Y \subset X$, a closed complex submanifold of dimension $m = n - k < n$. We have defined $\widetilde{X}_Y$ the blow-up of X along Y with the blow-down map $\pi : \widetilde{X}_Y \to X$ (in Exercises sheet 1).

(1) We denote the inclusion $\iota : Y \hookrightarrow X$. Show that the normal bundle $N_{Y/X} := \iota^*TX/TY$ is a holomorphic bundle over Y (it is a standard fact that the quotient of a holomorphic vector bundle by its subbundle is still holomorphic).
(2) Let $U \subset X$ be an open set such that $U \cap Y = Z(f_1^U) \cap \cdots \cap Z(f_k^U)$ where $f^U = (f_1^U, \cdots, f_k^U) : U \to \mathbb{C}^k$ is holomorphic (and 0 is a regular value). Show that $\{df_i\}_{i=1,\cdots,k}$ gives a frame of the dual bundle of the normal bundle over $U \cap Y$.
(3) Assume $k = 1$. Show that there is a line bundle $L \to X$ which is trivial on $X \setminus Y$ and agrees with the dual bundle of the normal bundle $N_{Y/X}$ over Y. (*Hint: use the transition functions.*)
(4) Show that $\pi^{-1}(Y) \simeq \mathbb{P}(N_{Y/X})$ as complex manifolds.
(5) Show that $\mathcal{O}_{\mathbb{P}(N_{Y/X})}(-1) \simeq N_{\pi^{-1}(Y)/\widetilde{X}_Y}$ as holomorphic bundle over $\pi^{-1}(Y)$.
(6) Convince yourself that $\pi^{-1}(Y)$ is a closed complex submanifold of complex codimension 1 in $\widetilde{X}_Y$. Using (3) show that there exists a line bundle $\mathcal{L} \to \widetilde{X}_Y$ whose restriction on $\pi^{-1}(Y)$ coincides with $\mathcal{O}_{\mathbb{P}(N_{Y/X})}(-1)$.
(7) Using (6), assuming (X, ω) is compact Kähler, show that there exists a hermitian metric h on $\mathcal{L}^{-1}$ and $\lambda > 0$ such that $F_{\nabla_h} + \lambda\pi^*\omega$ is a Kähler metric on $\widetilde{X}_Y$.