EXERCISES 4

Exercises for you to practice, think, and read something – no need to return.

1. Sheaves and Čech cohomology

Exercise 1. Let $\mathscr{A}_M^{p,q}$ be the sheaf of smooth (p,q)-forms on a compact complex manifold M. Show that $\check{H}^i(M, \mathscr{A}_M^{p,q}) = 0$ for all $i > 0$.

Exercise 2. Let $N \subset M$ be a complex submanifold and $\mathscr{F}$ a sheaf over N. Show that $U \mapsto \mathscr{F}_N(U) := \mathscr{F}(U \cap N)$ defines a sheaf over M.

Exercise 3. Verify the following properties regarding sheafification:

(1) Check that sheafification of a presheaf is a sheaf.
(2) On a complex manifold, check that the sheafification of the image of $\exp(2\pi i \bullet) : \mathcal{O} \to \mathcal{O}^*$ is $\mathcal{O}^*$.

Exercise 4. Let M be a compact complex manifold.

(1) Show that $\check{H}^1(M, \mathcal{O}^*)$ encodes the isomorphic classes of holomorphic line bundles on M.
(2) Show that

$$0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{e^{2\pi i \cdot}} \mathcal{O}^* \to 1$$

is a short exact sequence of sheaves, and check that it induces the following long exact sequence

$$\begin{array}{l} 0 \longrightarrow \check{H}^0(M,\mathbb{Z}) \longrightarrow \check{H}^0(M,\mathcal{O}) \longrightarrow \check{H}^0(M,\mathcal{O}^*) \xrightarrow{\delta^*} \\ \check{H}^1(M,\mathbb{Z}) \longrightarrow \check{H}^1(M,\mathcal{O}) \longrightarrow \check{H}^1(M,\mathcal{O}^*) \xrightarrow{\delta^*} \\ \check{H}^2(M,\mathbb{Z}) \longrightarrow \check{H}^2(M,\mathcal{O}) \longrightarrow \check{H}^2(M,\mathcal{O}^*) \longrightarrow \cdots \end{array}$$

(3) Show that the composition of the following maps

$$\check{H}^1(M,\mathcal{O}^*) \xrightarrow{\delta^*} \check{H}^2(M,\mathbb{Z}) \hookrightarrow \check{H}^2(M,\mathbb{R}) \simeq H^2_{dR}(M,\mathbb{R})$$

corresponds to the first Chern class of holomorphic line bundles.

Exercise 5 (Dolbeault theorem)**.** Let M be a compact complex manifold and let E be a holomorphic vector bundle on M. Denote by Ω^p the sheaf of holomorphic p-forms on M, and $\Omega^p(E)$ the sheaf of E-valued holomorphic p-forms on M. Prove that the Čech cohomology group $\check{H}^q(M, \Omega^p(E))$ is isomorphic to the Dolbeault cohomology group $H^{p,q}(M, E)$.

Exercise 6. Let Ω^p be the sheaf of holomorphic p-forms over $\mathbb{P}^n$. Show that

$$\check{H}^q(\mathbb{P}^n, \Omega^p) \simeq \begin{cases} \mathbb{C} & \text{if } q = p \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 7. Let $M = \mathbb{P}^n$ and $p, q \in M$ distinct points on M. Let $\mathcal{O}(-p-q)$ denote the sheaf of holomorphic functions on M vanishing at both p and q. Show that there is a short exact sequence of sheaves

$$0 \to \mathcal{O}(-p-q) \to \mathcal{O} \to \mathbb{C}_p \oplus \mathbb{C}_q \to 0$$

where the sheaves on the right-hand side should be carefully defined. Show that the map $\check{H}^0(M, \mathcal{O}) \to \check{H}^0(M, \mathbb{C}_p \oplus \mathbb{C}_q)$ is not surjective and conclude that $\check{H}^1(M, \mathcal{O}(-p-q)) \neq 0$.

Exercise 8. Show that any holomorphic line bundle on a disk is trivial. Deduce that any holomorphic line bundle on $\mathbb{P}^1$ is of the form $\mathcal{O}(n)$ for some integer n.

Exercise 9. Check that $\check{H}^q(\mathbb{C}^n, \mathcal{O}) = 0$ and $\check{H}^q(\mathbb{C}^n, \mathbb{Z}) = 0$ for $q > 0$. Using the exponential sheaf short exact sequence, deduce that $\check{H}^q(\mathbb{C}^n, \mathcal{O}^*) = 0$ for $q > 0$. Then conclude that an analytic hypersurface in $\mathbb{C}^n$ is the zero locus of an entire function.

2. Some analysis

Some references for you:

- Gilbarg–Trudinger, *Elliptic partial differential equations of second order*
- Aubin, *Some Nonlinear Problems in Riemannian Geometry*

Exercise 10. Let (M, J, ω) be an n-dimensional compact Kähler manifold. For any function $\varphi \in C^2(M)$, we define

$$\Delta_\omega \varphi \underset{loc}{:=} \sum_{1\le\alpha,\beta\le n} g^{\alpha\bar{\beta}} \frac{\partial^2\varphi}{\partial z_\alpha \partial \bar{z}_\beta}$$

where $\omega \underset{loc}{=} \sum_{1\le\alpha,\beta\le n} g_{\alpha\bar{\beta}} \mathrm{i} \mathrm{d}z_\alpha \wedge \mathrm{d}\bar{z}_\beta$.

(1) Show that Δ_ω is a second-order elliptic operator
(2) Show that $\Delta_\omega = -c\Delta_{\bar\partial}$ for some $c > 0$.
(3) Show that
$$\Delta_\omega \varphi = \frac{n\mathrm{i}\partial\bar\partial\varphi \wedge \omega^{n-1}}{\omega^n}.$$
(4) Show that if φ is a C^2 function such that $\Delta_\omega\varphi = 0$, then φ is constant.

Exercise 11. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $\varphi \in C^2(\Omega)$. Show that if $\Delta\varphi \ge 0$ (subharmonic), then φ satisfies the sub-mean value inequality:

$$\varphi(x) \le \frac{1}{|B(x,r)|} \int_{B(x,r)} \varphi(y)\mathrm{d}y$$

for all $x \in \Omega$ and $0 < r < \mathrm{dist}(x, \partial\Omega)$.

Exercise 12. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain.

(1) Show that for any $f \in C^{k,\alpha}(\Omega)$, f is a C^k function.
(2) Show that $C^{k,\alpha}(\Omega)$ is complete (i.e. A Cauchy sequence in $C^{k,\alpha}(\Omega)$ is a converging sequence in $C^{k,\alpha}(\Omega)$).
(3) Suppose that $(f_k)_{k\in\mathbb{N}}$ is a sequence in $C^{k,\alpha}(\Omega)$. Assume that there exist $M > 0$ such that $\|f_k\|_{C^{k,\alpha}(\Omega)} \le M$ for all k. Using Arzela–Ascoli theorem to show that for any $(\ell, \beta) \in \mathbb{N} \times (0, 1)$ such that $\ell + \beta < k + \alpha$, there exists a subsequence $(f_{k_j})_{j\in\mathbb{N}}$ converging in $C^{\ell,\beta}(\Omega)$.

Exercise 13. Let Ω be a bounded domain in $\mathbb{R}^n$.

(1) (Sobolev inequality) Show that for $1 \le p < n$, there exists a constant $C_S > 0$ such that for all $f \in C_c^\infty(\Omega)$,
$$\|f\|_{L^{\frac{np}{n-p}}(\Omega)} \le C_S \|\nabla f\|_{L^p(\Omega)}.$$
(2) (Poincaré inequality) Show that for all $1 \le p < +\infty$, there exists a constant $C_P > 0$ such that for all $f \in C_c^\infty(\Omega)$,
$$\|f\|_{L^p(\Omega)} \le C_P \|\nabla f\|_{L^p(\Omega)}$$
(3) (Poincaré–Wirtinger inequality) Show that for all $1 \le p < +\infty$, there exists a constant $C'_P > 0$ such that for all $f \in C^\infty(\Omega)$,
$$\|f - f_\Omega\|_{L^p(\Omega)} \le C'_P \|\nabla f\|_{L^p(\Omega)}$$
where $f_\Omega := \frac{1}{|\Omega|}\int_\Omega f(x)\mathrm{d}x$.
(4) Think about the version of the above inequalities on compact Riemannian manifolds.

Exercise 14 (Elliptic regularity theory)**.** Let (M, g) be a compact oriented Riemannian manifold and let L be a second-order elliptic operator with smooth coefficients (not necessarily self-adjoint). Show that for all $(k, \alpha) \in \mathbb{N} \times (0, 1)$,

$$L : (\ker L)^{\perp_{L^2}} \cap C^{k+2,\alpha}(M) \to (\ker L^*)^{\perp_{L^2}} \cap C^{k,\alpha}(M)$$

is an isomorphism, where L^* is the adjoint operator of L, i.e. $\int L(f_1)f_2 \,\mathrm{dvol}_g = \int_M f_1 L^*(f_2) \,\mathrm{dvol}_g$.