Orbifold Regularity of singular KE metrics

§1. Definitions & Setting & Standard results

Let X be a compact complex space. (in the sense of Grauert)

Defn: A Kähler metric ω_X on X_{reg} is called a "smooth Kähler metric on X" if $\forall x \in X$, $\exists U \ni x$ nbd & a holo embedding $j_x : U \hookrightarrow \mathbb{C}^{N_x}$ s.t. $\omega_X = j_x^* \theta$, where θ is a Kähler metric defined on a nbd of $j_x(U)$.

We call (X, ω_X) is a compact Kähler space. if ω_X is a smooth Kähler met.

Now, we further assume that X has at most "~~KLT~~ log-terminal singularities"

This means:

1) X is normal, roughly speaking $\mathcal{O}_{X_{reg}} \simeq \mathcal{O}_X$
   $\Rightarrow$ X is locally irreducible & X_{sing} has $\mathrm{codim}_{\mathbb{C}} \geq 2$.
2) K_X is a $\mathbb{Q}$-line bundle. ($\mathbb{Q}$-Gorenstein)
   i.e. $\exists m \in \mathbb{N}^*$ st. $mK_{X_{reg}} = L|_{X_{reg}}$ for some holo l.b. L on X.
3) Let $|\cdot|_{h^m}$ be a smooth hermitian metric on mK_X.
   $\mu_{h^m} := i^{n^2} \left(\dfrac{\sigma \wedge \bar\sigma}{|\sigma|^{2/m}_{h^m}} \right)^{1/m}$ the adapted measure
   lt $\Longleftrightarrow$ μ_{h^m} has finite mass.

Example: $A = \{ x_1^2 + \cdots + x_{n+1}^2 = 0 \} \subset \mathbb{C}^{n+1}$

Rmk: $\exists \pi : Y \to X$ log resolution
$K_Y = \pi^* K_X + \sum_i a_i E_i$, $a_i > -1$.

Defn: • $\mathrm{Ric}(\mu_{h^m}) := dd^c \log |\sigma|^{2/m}_{h^m}$ (it is smooth!)
• For a closed positive $(1,1)$-current T st. $T^n = f\mu$, $f \in L^1(\mu)$, we define $\mathrm{Ric}(T) = \mathrm{Ric}(\mu) - dd^c \log f$.

Under the above setting, a singular Kähler Einstein metric $\omega_{KE} \in [\omega_X]$ is a closed positive $(1,1)$-current $\omega_{KE} = \omega + dd^c\varphi$, $\varphi \in PSH(X,\omega) \cap L^\infty(X)$
st. • ω_{KE} is a smooth Kähler metric on X_{reg}.
• $\mathrm{Ric}(\omega_{KE}) = \lambda \omega_{KE}$, $\lambda \in \{0, \pm 1\}$
$\hookrightarrow$ $(\omega_X + dd^c\varphi)^n = e^{-\lambda\varphi}\mu$ $\begin{cases} \lambda = \pm 1, & \mathrm{Ric}(\mu) = \mp \omega_X \\ \lambda = 0, & \mathrm{Ric}(\mu) = 0 \end{cases}$

Rmk: $\lambda = -1, 0$: Eyssidieux-Guedj-Zeriahi '09 (sing KE always exist)
$= +1$: Chi Li, ($\exists$ sing KE $\Leftrightarrow$ K-stability)

Note: if $(X,x) \simeq (\mathbb{C}^n/G, 0)$ for some finite subgp $G \subset GL(n,\mathbb{C})$ then (X,x) is ~~klt~~.

- $0 \in (\sum_{i=1}^{n+1} z_i^2 = 0) \subset \mathbb{C}^{n+1}$ is ~~klt~~, but not quotient.
- (GKKP'11) X: compact Kähler w/ klt sing.

(Greb-Kebekus-Kovács-Peternell)

$\Rightarrow \exists\ Z \subset X$ an analytic subset of codim ≥ 3 s.t. $X \setminus Z$ has at most quotient singularities (moreover if $x \in X_{sing} \setminus Z \Rightarrow (X,x) \simeq S \times (\mathbb{C}^{n-2}, 0)$)

Question: Let X_{orb} be euclidean open subset of X where X has quotient sing.

$\forall\ x \in X_{orb}$, $\exists$ local uniformizing chart $p: V \to \mathcal{U} \ni x$ where V is smooth:

Does $p^*(\omega_{KE}|_{\mathcal{U}_{reg}})$ extend to a smooth Kähler metric on V?

§2. Results.

Thm (Guenancia-Păun'24): ω_{KE} sing KE on (X, ω_X) compact Kähler w/ klt singularities

(1) $\exists\ Z$ codim ≥ 3 st. ω_{KE} is an orbifold Kähler metric on $X \setminus Z$.

(2) ω_{KE} is strictly positive on $X \setminus Z$

i.e. $\forall\ \mathcal{U} \Subset X \setminus Z$, $\exists\ \delta(\mathcal{U}) > 0$ st. $\omega_{KE} \geq \delta(\mathcal{U})\, \omega_X$ on $\mathcal{U}$

(next time) Thm (Székelyhidi-co'25) ω_{KE} —, (X, ω_X) —

ω_{KE} is strictly positive on X.

i.e. $\exists\ \delta > 0$ st. $\omega_{KE} \geq \delta\, \omega_X$ on X.

Usual strategy to get smoothness:

- L^∞-estimate (global) $\longrightarrow$ okay by EGZ.
- Δ-estimate (global) $\longrightarrow$???
- Evans-Krylov ($L^\infty + \Delta \to C^{2,\alpha}$, local), Schauder ($C^{2,\alpha} \to$ higher, local) $\longrightarrow$ okay

(admitted first) Prop: Let $\mathcal{U} \subset X$ be a domain, $\mathcal{U}^{sing}$ admits a family of cutoffs $(\rho_\delta)_{\delta>0}$ st. $\lim_{\delta\to 0} \int_{\mathcal{U}} |\nabla \rho_\delta|_\omega^{2+\varepsilon_0} + |\Delta_\omega \rho_\delta|^{1+\varepsilon_0}\, \omega^n = 0$ for some $\varepsilon_0 > 0$

Then $\exists\ p_0 > 0$ st. $\forall\ f \in C^2(\mathcal{U}_{reg})$, $\mathrm{Supp}(f) \subset \mathcal{U}$ w/ $f \in L^{p_0}(\mathcal{U})$ & $\Delta_\omega f \geq -g$ on $\mathcal{U}_{reg}$, for some $g \in L^{p_0}(\mathcal{U})$

$\Longrightarrow \sup_{\mathcal{U}_{reg}} f < +\infty$

§3. Proof of Thm (GP) (1)

Assume that $\omega_{KE} > \delta\omega_X$ for some $\delta > 0$.

The statement is local,

Fix $x \in X_{orb}$, 2 open nbd $\overset{x}{\overset{\cap}{\mathcal{U}}} \Subset \mathcal{U}' \Subset X_{orb}$

and $p: V' \to \mathcal{U}'$ ~~surjective finite cover~~ (quasi-étale)

s.t. V': nbd of 0 in $\mathbb{C}^n$

and $\mathcal{U}' \simeq V'/G \leftarrow$ finite subgp of $GL(n, \mathbb{C})$

$$W' = p^{-1}(\mathcal{U}'^{sing}) = \bigcup_{g \in G \setminus \{e\}} Fix(g)$$

Set $\hat{\omega} = p^*\omega_{KE}$

$\uparrow$ positive current on V' with bdd potential

smooth on $V' \setminus W'$.

$Ric(\hat{\omega}) \geq -C\hat{\omega}$

Let η be a non-vanishing holo 1-form on V' (e.g. dz_1)

and define $f = |\eta|^2_{\hat{\omega}}$ (smooth on $V' \setminus W'$)

(if we can bound f from above $\to$ done
how: use Prop)

Note that $p: V' \to \mathcal{U}'$ is quasi-étale $\Rightarrow K_{V'} = p^*K_{\mathcal{U}'}$

and hence $p^*\mu \simeq dV_{\mathbb{C}^n}$

- $f \in L^1(V', dV_{\mathbb{C}^n})$:

$$\int_{V'} f\, dV_{\mathbb{C}^n} \simeq \int_{V'} f\, p^*\mu \simeq \int_{V'} f\,\hat{\omega}^n = \int_{V'} \eta \wedge \bar{\eta} \wedge \hat{\omega}^{n-1} \underset{\uparrow\ \hat{\omega}\text{ has bdd potential}}{<} +\infty$$

$\hookrightarrow$ will use f^ε as the fcn in Prop for some $\varepsilon \in (0,1)$

- f is not compact support, need some cutoff fcns

$\hookrightarrow$ next page

Consider $\chi \geq 0$ a smooth cutoff on X st $\begin{cases} \chi \equiv 1 \text{ on } \mathcal{U} \\ \mathrm{supp}(\chi) \subset \mathcal{U}' \end{cases}$

and $\begin{cases} |\nabla\chi|^2_{\omega_X} \leq C\chi \\ |dd^c\chi|^2_{\omega_X} \leq C \end{cases} \underset{\omega_{KE} \geq \delta\omega_X}{\Longrightarrow} \begin{cases} |\nabla\chi|^2_{\hat\omega} \leq C\chi \\ |dd^c\chi|^2_{\hat\omega} \leq C. \end{cases}$

↳ χf^ε is $C^2(V'\setminus W')$, compact supp on V'.

- Laplacian estimate:

$$\Delta_{\hat\omega}(\chi f^\varepsilon) = \chi \underbrace{\Delta_{\hat\omega} f^\varepsilon}_{①} + \underbrace{\langle \nabla\chi, \nabla f^\varepsilon\rangle_{\hat\omega}}_{②} + f^\varepsilon \Delta_{\hat\omega}\chi.$$

$$① = \varepsilon f^{\varepsilon-1} \Delta_{\hat\omega} f - \varepsilon(1-\varepsilon) f^{\varepsilon-2} \mathrm{tr}_{\hat\omega}(\partial f \wedge \bar\partial f)$$

$$\hookrightarrow = |\nabla\eta|^2_{\hat\omega} + \langle \mathrm{Ric}^\#(\hat\omega)\cdot\eta, \eta\rangle_{\hat\omega}$$

$$\boxed{\mathrm{Ric}\,\hat\omega \geq -C\hat\omega} \to\ \geq |\nabla\eta|^2_{\hat\omega} - C|\eta|^2_{\hat\omega} = |\nabla\eta|^2_{\hat\omega} - Cf$$

$$\to \mathrm{tr}_{\hat\omega}(\partial f \wedge \bar\partial f) \leq |\langle \nabla\eta, \eta\rangle_{\hat\omega}|^2_{\hat\omega} \leq |\nabla\eta|^2_{\hat\omega}|\eta|^2_{\hat\omega}$$

$$\Rightarrow ① \geq \varepsilon^2 f^{\varepsilon-1}|\nabla\eta|^2_{\hat\omega} - \varepsilon C f^\varepsilon$$

$$② = \varepsilon f^{\varepsilon-1}\langle\nabla\chi, \nabla f\rangle_{\hat\omega} \leq \varepsilon f^{\varepsilon-1}|\nabla\chi|_{\hat\omega}\cdot\overbrace{|\eta|_{\hat\omega}}^{f^{1/2}}|\nabla\eta|_{\hat\omega}$$

$$\leq \delta\varepsilon f^{\varepsilon-1}|\nabla\chi|^2_{\hat\omega}|\nabla\eta|^2_{\hat\omega} + \delta^{-1}\varepsilon f^\varepsilon$$

$$\leq \delta\varepsilon C\chi f^{\varepsilon-1}|\nabla\eta|^2_{\hat\omega} + \delta^{-1}\varepsilon f^\varepsilon$$

$$\Delta_{\hat\omega}(\chi f^\varepsilon) \geq (\varepsilon - \delta C)\varepsilon\chi f^{\varepsilon-1}|\nabla\eta|^2_{\hat\omega} - (C\varepsilon + \delta^{-1}\varepsilon + C)f^\varepsilon$$

take $\delta = \varepsilon/C$ $\to\ \geq -C'\cdot f^\varepsilon$.

make it G-inv: $\sum_{g\in G} g\cdot(\chi f^\varepsilon) = F$ positive G-inv, dominate χf^ε

$F \in L^{1/\varepsilon}$ and $\Delta_{\hat\omega} F \geq -g \in L^{1/\varepsilon}$ $\underset{\text{prop}}{\Longrightarrow}$ $F < +\infty \Rightarrow f < +\infty$

$\Rightarrow \mathrm{tr}_{\hat\omega}\,\omega_\varepsilon^n < +\infty$. (need to construct f_s later)

- higher regularity: Evans Krylov: $\hat{\varphi} \in C^{2,\alpha}(V \setminus W')$ (no 'loc') — potential of $\hat{\omega}$

  $\Rightarrow \hat{\varphi} \in C^{2,\alpha}(V) \underset{\text{Schauer}}{\Longrightarrow} \hat{\varphi} \in C^{\infty}(V')$

- Constructing cutoff $(\rho_\delta)_\delta$: (some problem in the argument).

Recall that we have $p : V' \to U'$, $W' = p^{-1}(U^{sing}) = \bigcup_{g \in G \setminus \{e\}} \mathrm{Fix}(g)$

note that we can assume G does not contains no "pseudo-reflections" (pr)

$\left(\begin{array}{l} 1 \neq g \in GL(n, \mathbb{C}) \text{ is pr if } \ker(g - \mathrm{Id}) \text{ has codim } 1, \\ \text{Chevalley - Shephard - Todd thm} : \mathbb{C}^n / G_{pr} \simeq \mathbb{C}^n \end{array} \right)$

$\Rightarrow W'$ is a union of codim ≥ 2 linear subsp.

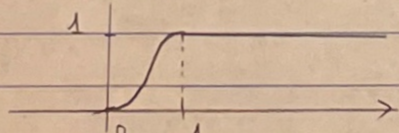
WLOG, we can assume that $W' = (z_1 = z_2 = 0) \cap V'$ and z_1, z_2 are G-invariant

↳ $\exists f_i$ holo on U' s.t. $p^* f_i = z_i$

Define $\psi = \log(|f_1|^2 + |f_2|^2)$ psh.

$$\Rightarrow dd^c\psi + d\psi \wedge d^c\psi \leq C \cdot e^{-\psi} \omega_X \leq C e^{-\psi} \omega_{KE}$$

strict positivity

Let ξ :  and $\rho_\delta(x) = \xi(\psi(x) + \frac{1}{\delta})$

$$\begin{cases} \rho_\delta = 1 & \text{on } \{\psi \geq 1 - 1/\delta\} \\ \rho_\delta \equiv 0 & \text{on } \{\psi < -1/\delta\} \end{cases}$$

$$|\Delta_{\omega_{KE}} \rho_\delta| = \left| \mathrm{tr}_{\omega_{KE}} (\xi' \cdot dd^c\psi + \xi'' d\psi \wedge d^c\psi) \right| \leq C e^{-\psi}$$

$$|\nabla \rho_\delta|^2_{\omega_{KE}} = \mathrm{tr}_{\omega_{KE}} (\xi')^2 d\psi \wedge d^c\psi \leq C e^{-\psi}$$

$$\Rightarrow \int_U |\Delta_{\omega_{KE}} \rho_\delta|^p + |\nabla \rho_\delta|^{2p}_{\omega_{KE}} \, \omega_{KE}^n \leq C \int_V \frac{1}{(|z_1|^2 + |z_2|^2)^p} dV_{\mathbb{C}^n} \leq \frac{C}{2-p}$$

ok for $1 < p < 2$ #