Orbifold Regularity of singular KE metrics II

§0. GPSS (Guo-Phong-Song-Sturm)

Let (X, ω_X) be an n-dim'l compact Kähler manifold.

$K(p, A, B) = \left\{ \omega \text{ Kähler metric on } X \;\middle|\; \begin{array}{l} [\omega]\cdot[\omega_X]^{n-1} \leq A,\ \|f_\omega\|_{L^p(\omega_X^n)} \leq B \\ \text{where } f_\omega = \dfrac{\omega^n}{V_\omega\, \omega_X^n} \end{array} \right\}$

$p > 1,\ A, B > 0$

- Heat kernel estimate:

heat kernel: $\begin{cases} \frac{\partial}{\partial t} H(x,y,t) = \Delta_{\omega,y} H(x,y,t) \\ \lim_{t\to 0^+} H(x,y,t) = \delta_x(y) \end{cases}$

Thm (GPSS): $\forall\, q \in (1, \frac{n}{n-1})$, $\exists\, C(n,p,A,B) > 0$ s.t. $\forall\, x, y \in X$, $\forall\, \omega \in K(p,A,B)$

$$H(x,y,t) \leq \begin{cases} \frac{C}{V_\omega}\left(\frac{A}{t}\right)^{\frac{q}{q-1}} \exp\left(-\frac{d_\omega(x,y)^2}{10t}\right) & \text{if } t \in (0, I_\omega] \\ \frac{C}{V_\omega} \exp\left(-\frac{d_\omega(x,y)^2}{10t}\right) & t \in (I_\omega, +\infty) \end{cases}$$

where $I_\omega = [\omega]\cdot[\omega_X]^{n-1}$

- Green's function estimate:

Green's function: $\begin{cases} \Delta_{\omega,y} G(x,y) = -\delta_x(y) + \frac{1}{V_\omega} \\ \int_X G(x,y)\, \omega^n(y) = 0 \end{cases}$

Thm (GPSS): $\forall\, r \in (0, \frac{n}{n-1})$, $\forall\, s \in (0, \frac{2n}{2n-1})$, $\forall\, \omega \in K(p,A,B)$

we have 1) $\inf_X G_x \geq -C_0(n,p,A,B)$

2) $\frac{1}{V_\omega}\int_X |G_x|^r\, \omega^n(y) \leq C_1(n,p,r,A,B)$

3) $\frac{1}{V_\omega}\int_X |\nabla G_x|^s\, \omega^n(y) \leq C_2(n,p,s,A,B)$.

- Diameter estimate & non-collapsing.

Thm (GPSS): $\forall\, \delta \in (0,1)$, $\forall\, \omega \in K(p,A,B)$, $\exists\, C(n,p,\delta,A,B) > 0$

s.t. $\mathrm{diam}(X,\omega) \leq C$ & $\mathrm{Vol}_\omega(B_\omega(x,r)) \geq \frac{V_\omega}{C}\cdot \min\{1, r^{2n+\delta}\}$

$\forall\, x \in X,\ r > 0$.

§1 Strict positivity of singular KE metrics

Let (X, ω_X) be a compact Kähler space with l.t. singularities

& $c_1(X) = \lambda[\omega_X]$, $\lambda \in \{\pm 1, 0\}$

Suppose that $\omega_{KE} = \omega_X + dd^c\varphi$ is the singular KE metric

i.e. $Ric(\omega_{KE}) = \lambda \omega_{KE}$

$\Leftrightarrow (\omega_X + dd^c\varphi)^n = e^{-\lambda\varphi}\mu$, where $Ric(\mu) = \lambda\omega_X$

Thm (Székelyhidi - co '25) $\exists\, \delta > 0$ s.t. $\omega_{KE} \geq \delta\omega_X$ on X

We consider the CY case for simplicity.

pf. Step 1) approximations:

Let $\pi: \hat{X} \to X$ be a log resolution.

$\pi^*\mu = \prod_i |s_i|^{2a_i} dV_{\hat{X}}$, $a_i > -1$, $dV_{\hat{X}}$: smooth volume form on $\hat{X}$

$(s_i = 0) = E_i$

$\downarrow$

$Ric(dV_{\hat{X}}) = \pi^*\underbrace{Ric(\mu)}_{=0} - \sum_i a_i \Theta(E_i)$

Take $\omega_{\hat{X}}$: Kähler metric on $\hat{X}$, $\omega_\varepsilon := \pi^*\omega_X + \varepsilon\omega_{\hat{X}}$

Consider $\underbrace{(\omega_\varepsilon + dd^c u_\varepsilon)}_{=: \hat{\omega}_\varepsilon}{}^n = c_\varepsilon \cdot \prod_i (|s_i|^2 + \varepsilon^2)^{a_i} dV$

- EGZ '09: $\|u_\varepsilon\|_{L^\infty} \leq C$ (indep of $\varepsilon \in [0,1]$ &

Step 2) Chern-Lu ineq. + heat kernel

$$\Delta_{\hat{\omega}_\varepsilon} \log tr_{\hat{\omega}_\varepsilon} \pi^*\omega_X \geq \frac{\hat{g}_\varepsilon^{i\bar{l}}\, \hat{g}_\varepsilon^{k\bar{j}}\, Ric(\hat{\omega}_\varepsilon)_{i\bar{j}}\, (g_X)_{k\bar{l}}}{tr_{\hat{\omega}_\varepsilon} \pi^*\omega_X} - 2B\, tr_{\hat{\omega}_\varepsilon} \pi^*\omega_X$$

where $B > 0$ s.t. $Bisec(\omega_X) \leq B$,

$Ric(\hat{\omega}_\varepsilon) = Ric(dV_{\hat{X}}) - \sum_i a_i dd^c \log(|s_i|^2 + \varepsilon^2)$

$= -\sum_i a_i (\Theta_i(E_i) + dd^c \log(|s_i|^2 + \varepsilon^2))$

$= \alpha - \beta$ where $\alpha = \sum_{a_i < 0} (-a_i)(\cdots)$, $\beta = \sum_{a_i > 0} a_i(\cdots)$

$\geq -|(\alpha)_- + (\beta)_+|_{\hat{\omega}_\varepsilon} - 2B\, tr_{\hat{\omega}_\varepsilon} \pi^*\omega_X$

$\Delta_{\hat{\omega}_\varepsilon}(\log tr_{\hat{\omega}_\varepsilon} \pi^*\omega_X - 2B u_\varepsilon) \geq -|\alpha_- + \beta_+|_{\hat{\omega}_\varepsilon} - 2B\, tr_{\hat{\omega}_\varepsilon} \pi^*\omega_X$

$- 2B\, tr_{\hat{\omega}_\varepsilon}(\omega_\varepsilon + dd^c u_\varepsilon) + 2B\, tr_{\hat{\omega}_\varepsilon} \omega_\varepsilon$ ≥ 0

$\geq |\alpha_- + \beta_+|_{\hat{\omega}_\varepsilon} - 2Bn.$

Let $F = \max\{0, \log \operatorname{tr}_{\hat{\omega}_\varepsilon} \pi^* \omega_X - 2B u_\varepsilon\}$

$\Rightarrow \Delta_{\hat{\omega}_\varepsilon} F \geq -|\alpha_- + \beta_+| - C.$

Fix $x \in \hat{X} \setminus E$ and let $H(x,y,t)$ be the heat kernel on $(\hat{X}, \omega_{\varepsilon,\delta})$.

[GPSS : $\exists \overline{H}(t) > 0$ s.t. $H(x,y,t) \leq \overline{H}(t)$ $\forall t \in (0,2]$, $\forall \varepsilon, \delta \in (0,1)$

and $\overline{H}(t) \to +\infty$ as $t \to 0^+$.

$$\partial_t \int F(y) H(x,y,t) \hat{\omega}_\varepsilon^n(y) = \int F(y) \Delta_{\omega_{\varepsilon,\delta}, y} H(x,y,t) \hat{\omega}_\varepsilon^n(y)$$

$$= \int \Delta_{\omega_\varepsilon} F\, H(x,y,t) \hat{\omega}_\varepsilon^n(y) \geq - \int |\alpha_- + \beta_+| H(x,y,t) \hat{\omega}_\varepsilon^n(y) - C$$

$$\geq - \overline{H}(t) \|\alpha_- + \beta_+\|_{L^1(\hat{\omega}_\varepsilon)} - C$$

Claim : $\|\alpha_- + \beta_+\|_{L^1(\hat{\omega}_\varepsilon)} \longrightarrow 0$ $\delta, \varepsilon \to 0$

$\Rightarrow \forall t_0 > 0$, $\exists \delta, \varepsilon$ small s.t. $\partial_t \int F(y) H(x,y,t) \hat{\omega}_\varepsilon^n(y) \geq -2C$.

$\Rightarrow \int F(y) H(x,y,t_0) \hat{\omega}_\varepsilon^n(y) \leq 2C + \int F(y) H(x,y,1) \hat{\omega}_\varepsilon^n(y) \leq C'$

$\log \operatorname{tr}_{\hat{\omega}} \omega_X \leq \operatorname{tr}_{\hat{\omega}} \omega_X$, $\int \operatorname{tr}_{\hat{\omega}} \omega_X < A$

Let $\varepsilon \to 0$ and then $t_0 \to 0$ $\Rightarrow$ $\operatorname{tr}_{\omega_{KE}} \omega_X \leq C$ #.

Pf of Claim :

Note that $\Theta(E) + dd^c \log(|s|^2 + \varepsilon^2) = \dfrac{\varepsilon^2}{|s|^2 + \varepsilon^2} \Theta(E) + \dfrac{\varepsilon^2 \langle D's, D's \rangle}{(|s|^2 + \varepsilon^2)^2}$

$$\geq \frac{\varepsilon^2}{|s|^2 + \varepsilon^2} \Theta(E) \geq -A \frac{\varepsilon^2}{|s|^2 + \varepsilon^2} \omega_{\hat{X}}$$

$$\alpha, \beta \geq \begin{cases} -A\omega_{\hat{X}} & \text{on } \hat{X} \\ -K\omega_{\hat{X}} & \text{on } \hat{X} \setminus N_K, \quad N_K \approx \bigcup_i \left\{ \dfrac{A\varepsilon^2}{|s_i|^2 + \varepsilon^2} > K \right\} : \text{nbd of } E \end{cases}$$

$$\Rightarrow |\alpha_-|_{\hat{\omega}_\varepsilon} \geq \begin{cases} -A \operatorname{tr}_{\hat{\omega}_\varepsilon} \omega_{\hat{X}} & \text{on } \hat{X} \\ -K \operatorname{tr}_{\hat{\omega}_\varepsilon} \omega_{\hat{X}} & \text{on } \hat{X} \setminus N_K \end{cases}$$

$$\int |\alpha_-|_{\hat\omega_\varepsilon} \hat\omega_\varepsilon^n \le \kappa \int_{\hat X} \omega_X \wedge \hat\omega_\varepsilon^{n-1} + A \int_{N_\kappa} \omega_{\hat X} \wedge \hat\omega_\varepsilon^{n-1} \xrightarrow{\kappa \to 0} 0$$

(exercise: $\int_{\{\varphi < -M\}} \omega_u^n \le \frac{1}{M} \underbrace{\int_X -\varphi\, \omega_u^n}_{\text{bdd}}$) ← ω-psh

On the other hand,

$$|\beta_+|_{\hat\omega_\varepsilon} \le \begin{cases} \mathrm{tr}_{\hat\omega_\varepsilon}(\beta + A\omega_{\hat X}) & \text{on } \hat X \\ \mathrm{tr}_{\hat\omega_\varepsilon}(\beta + \kappa\omega_{\hat X}) & \text{on } \hat X \setminus N_\kappa \end{cases}$$

$$\Rightarrow \int |\beta_+|_{\hat\omega_\varepsilon} \omega_\varepsilon^n \le \underbrace{\int_{\hat X} \beta \wedge \hat\omega_\varepsilon^{n-1}}_{\| } + \underbrace{\kappa \int_{\hat X} \omega_{\hat X} \wedge \hat\omega_\varepsilon^{n-1} + A \int_{N_\kappa} \omega_{\hat X} \wedge \hat\omega_\varepsilon^{n-1}}_{\xrightarrow[\kappa \to 0]{} 0}$$

$$\sum_{a_i > 0} a_i \int_{E_i} \omega_\varepsilon^{n-1} \xrightarrow[\varepsilon \to 0]{} 0 \qquad \#$$

§2) A mean value inequality. $X, \omega_X, \omega_{KE} = \omega$ as usual.

Prop: Assume that X_{sing} admits a family of cutoffs $(\rho_\delta)_{\delta > 0}$ s.t.

$$\int_X |\nabla \rho_\delta|_\omega^{2+\varepsilon} + |\Delta_\omega \rho_\delta|^{1+\varepsilon} \omega^n \xrightarrow[\delta \to 0]{} \quad \text{for some } \varepsilon > 0.$$

Then $\exists\, p_0 > 0$ st. $\forall\ 0 \le f \in C^2(X_{reg})$ satisfying

$$\begin{cases} f \in L^{p_0}(X, \omega^n) \\ \Delta_\omega f \ge -g \text{ on } X_{reg} \text{ for some } g \in L^{p_0}(X, \omega^n) \end{cases}$$

$\Longrightarrow \sup_{X_{reg}} f < +\infty$

Recall that we have approximations $\hat\omega_\varepsilon := \omega_\varepsilon + dd^c u_\varepsilon$ on $\hat X$

$\hat X \xrightarrow{\pi} X$

which solves $(\omega_\varepsilon + dd^c u_\varepsilon)^n = c_\varepsilon \prod_i (|s_i|^2 + \varepsilon^2)^{a_i} dV_{\hat X}$.

and $\hat\omega_\varepsilon \xrightarrow[C^\infty_{loc}(\hat X \setminus E)]{\varepsilon \to 0} \omega$.

GPSS: ① $\exists\, C, \delta > 0$ indep of ε st. $\forall x \in \hat X$

$$\int |G_\varepsilon(x,\cdot)|^{1+\delta} + |\nabla G_\varepsilon(x,\cdot)|^{1+\delta}\, \hat\omega_\varepsilon^n \le C.$$

② $\forall \beta > 0$. $\exists\, C_\beta > 0$ indep of ε st. $\forall x \in \hat X$, $\int \frac{|\nabla G_\varepsilon(x,\cdot)|^2}{|G_\varepsilon(x,\cdot)|^{1+\beta}} \hat\omega_\varepsilon^n \le C_\beta$.

Fix $x \in \hat{X} \setminus E \simeq X_{reg}$

We have unif $W^{1,1+\delta}_{loc}$ bounds of $G_\varepsilon(x,\cdot)$ on X_{reg}

By std Sobolev embedding, $\exists$ subseq of $G_\varepsilon(x,\cdot) \xrightarrow[L^1_{loc}]{} G_x$

(By Fatou + ⋯) one can check

- $\inf_{X_{reg}} G_x \geq -C$
- $\Delta_\omega G_x = -\delta_x + \frac{1}{V_\omega}$ weakly on X_{reg}.
- $\int_{X_{reg}} |G_x|^{1+\delta} + |\nabla G_x|^{1+\delta} \omega^n \leq C$
- $\forall \beta > 0 \quad \int_X \frac{|\nabla G_x|^2}{|G_x|^{1+\beta}} \leq C_\beta.$

these constants are indep of $x \in X_{reg}$

Rmk: G_x is smooth on $X_{reg} \setminus \{x\}$, indeed $\Delta_\omega G_x = -\frac{1}{V}$ there.

Proof of Prop.

By decreasing $\varepsilon > 0$, we have

$$\int |G_x|^{1+\varepsilon} + |\nabla G_x|^{1+\varepsilon} + \frac{|\nabla G_x|^2}{|G_x|^{1+\varepsilon}} \omega^n \leq C \quad \forall x \in X_{reg}$$

Let p, p' be the conjugate exponent of $1+\varepsilon$, $1+\varepsilon/2$

i.e. $1 = \frac{1}{p} + \frac{1}{1+\varepsilon}$ — —

take $p_0 > \max\{p, 2p'\}$

Fix $x \in X_{reg}$, we claim: $f(x) \leq \underbrace{\int f \omega^n + \|G_x\|_{L^{1+\varepsilon}} \|g\|_{L^{p_0}}}_{=: M} \xrightarrow[\text{on } f]{+\text{ assumption}}$ done

If we can prove: $\forall\, 0 \leq h \in C^\infty_c(X_{reg})$, $\int h(x) f(x) \omega^n \leq M \int h(x) \omega^n$

$\Rightarrow$ Claim is okay.

First consider $\rho_\delta f \in C^2_c(X_{reg})$

$\hookrightarrow \rho_\delta f(x) = \int \rho_\delta f \omega^n - \int G_x \Delta_\omega(\rho_\delta f) \omega^n$

$$-\int G_x \Delta_\omega(\rho_\delta f)\omega^n = -\int G_x \rho_\delta \Delta_\omega f \omega^n - \int G_x f \Delta_\omega \rho_\delta \omega^n - 2\overbrace{\int G_x \langle \nabla \rho_\delta, \nabla f \rangle \omega^n}^{\substack{-\int f \langle \nabla G_x, \nabla \rho_\delta \rangle \omega^n \\ -\int f G_x \Delta_\omega \rho_\delta \omega^n}}$$

$$= \underbrace{-\int G_x \rho_\delta \Delta_\omega f \omega^n}_{①} + \underbrace{\int G_x f \Delta_\omega \rho_\delta \omega^n}_{②} + 2\underbrace{\int f \langle \nabla G_x, \nabla \rho_\delta \rangle \omega^n}_{③}$$

$|①| \leq \|G_x\|_{L^{1+\varepsilon}} \|g\|_{L^p}$

$$② : \left| \int_x h(x) \left(\int_y f(y)\, G_x(y)\, \Delta_\omega \rho_\delta(y)\, \omega^n(y) \right) \omega^n(x) \right|$$

$$\leq \left| \int_y f(y)\, \Delta_\omega \rho_\delta(y) \underbrace{\int_x |h(x)\, G_y(x)|\, \omega^n(y)}_{\leq \sup|h| \cdot \|G_y\|_{L^1}} \omega^n(x) \right|$$

$$\leq \sup|h| \cdot \|G_y\|_{L^1} \cdot \|f\|_{L^p} \|\Delta_\omega \rho_\delta\|_{L^{1+\varepsilon}} \xrightarrow[\delta \to 0]{} 0$$

$$③ : \left| \int f \langle \nabla G_x, \nabla \rho_\delta \rangle \omega^n \right| \leq \left(\int f^2 |G_x|^{1+\varepsilon} |\nabla \rho_\delta|^2 \omega^n \right)^{1/2} \left(\int \frac{|\nabla G_x|^2}{|G_x|^{1+\varepsilon}} \omega^n \right)^{1/2}$$

$$\leq C \cdot \left(\int f^2 |G_x|^{1+\varepsilon} |\nabla \rho_\delta|^2 \omega^n \right)^{1/2}$$

$$\to \left| \int_x h(x) \left(\int_y f(y) \langle \nabla G_x, \nabla \rho_\delta \rangle(y)\, \omega^n(y) \right) \omega^n(x) \right|$$

$$\leq C \cdot \left(\int_y f^2(y)\, |\nabla \rho_\delta|^2(y) \left(\int_x h^2(x)\, |G_y(x)|^{1+\varepsilon} \omega^n(x) \right) \omega^n(y) \right)^{1/2}$$

$$\leq C \sup h^2 \cdot \|G_y\|_{L^{1+\varepsilon}}^{\frac{1+\varepsilon}{2}} \int_y f^2(y)\, |\nabla \rho_\delta|^2(y)\, \omega^n(y)$$

$$\leq \text{———} \cdot \|f\|_{L^{2p'}}^2 \cdot \|\nabla \rho_\delta\|_{L^{1+\varepsilon}}^2 \xrightarrow[\delta \to 0]{} 0$$

All in all:

$$\int h(x)\, \rho_\delta f(x)\, \omega^n = \int_x h(x) \cdot \left(\int \rho_\delta f \omega^n \right) \omega^n(x) - \int h \cdot (① + ② + ③)\, \omega^n$$

$$\downarrow \delta \to 0 \qquad ② \to 0, \quad ③ \to 0$$

$$\int h \cdot f \omega^n \leq \left(\int h\, \omega^n \right) \cdot \left(\int f \omega^n + \|G_x\|_{L^{1+\varepsilon}} \|g\|_{L^p} \right)$$

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