① Setting:

- X: irreducible, reduced complex space
- $\pi : X \longrightarrow 2\mathbb{D} \subset \mathbb{C}$
  Proper, surjective, holomorphic
- $X_t := \pi^{-1}(\{t\})$ is a smooth compact Kähler for $t \neq 0$
  and allow X_0 to be singular
- β a smooth relative

Kähler form. i.e.

$\beta_t := \beta|_{X_t}$ is a Kähler form

- $V_{\beta_t} := \int_{X_t} \beta_t^n$ are uniformly bounded away from 0 and ∞.

$$dV_{X_t} := \frac{\beta_t^n}{V_{\beta_t}} \to \text{Probability measure}$$

- Fix $p > 1$, $A, B > 0$

$\mathcal{K}(X, p, A, B) :=$ w relative

Kähler form on $\mathcal{X} \setminus X_0$ s.t.:

- $[\omega_t] \leqslant A\,[\beta_t]$ in $H^{1,1}(X_t, \mathbb{R})$

(Uniform bound of Kähler classes) $\Rightarrow$ $\boxed{V_{\omega_t} \leqslant C(A)}$

- If $\dfrac{\omega_t^n}{V_{\omega_t}} = f_t \, dV_{X_t}$

then $$\int_{X_t} f_t^p \, dV_{X_t} \leqslant B$$

(L^p-uniform bound of Volumes)

Goal: G^{ω_t} Green function

Goal: G_x : Green function

① $\sup_{x_t} G_x^{\omega_t} \leq C_0(n,p,A,B)$

(L^∞-uniform bound)

② $\frac{1}{V_{\omega_t}} \int |G_x^{\omega_t}|^r \omega_t^n \leqslant C_1(n,p,r,A,B)$

for $r \in \left(0, \frac{n}{n-1}\right)$

(L^r-uniform bound)

③ $\frac{1}{V_{\omega_t}} \int |\nabla G_x^{\omega_t}|^s \omega_t^n \leqslant$

$C(p,n,s,A,B)$

$C_2(\ldots)$

for $s \in \left(0, \frac{2n}{2n-1}\right)$

(L^s-uniform bound for the gradient)

Application:

1 For $\delta \in (0,1)$, $\exists\, C(n,p,\delta,A,B) > 0$:

- $\dfrac{\mathrm{Vol}_{\omega_t}(B_{\omega_t}(x,r))}{V_{\omega_t}} \geq C \min(1, r^{2n+\delta})$

(non-collapsing)

- $\mathrm{diam}(X, \omega_t) \leq C$

- diam$(X_t, t) \leq$

(uniform bound of diameter)

$$\xrightarrow[\text{volume bound}]{\text{+ uniform}} \text{G-H}$$

pre-compact.

② Applied to Kähler-Ricci flow:

X, K_X nef, ω_0 fixed

$$\frac{\partial \omega_t}{\partial t} = -Ric(\omega_t) - \omega_t$$

$\Longrightarrow$ flow exists $\forall t > 0$

$\Longrightarrow$ For $\delta \in (0,1)$, $\exists C(\omega_0) > 0$ and $c(\omega_0, \delta) > 0$ s.t :

- $\operatorname{diam}(X, \omega_t) \leq C$
- $\operatorname{Vol}_{\omega_t}(B_{\omega_t}(x,r)) \geq c\, r^{2n+\delta}\, V_{\omega_t}$

for $r \leq \operatorname{diam}(X, \omega_t)$

Uniform Green-function estimates

Reminder :

① (M, g) Riemannian mfd

- $f : M \longrightarrow \mathbb{R}$ upper semi-cont
- $\forall$ open $U \subset M$, $\tilde{f}$ harmonic on U

  $\tilde{f} \geq f$ on $\partial U \Rightarrow \tilde{f} \geq f$ on U

  $\Rightarrow f$ is subharmonic

- f is C^2 :

  subharmonic $\Longleftrightarrow \Delta f \geq 0$

- $f : \mathbb{C} \longrightarrow \mathbb{R}$ subharmonic

  $\Longleftrightarrow \forall r : \quad \varphi(z) \leq \frac{1}{2\pi}\int_0^{2\pi} \varphi(z + re^{i\theta})$

② (X, J) complex mfd.

$f : X \longrightarrow \mathbb{R} \cup \{-\infty\}$ upper-semi continuo

- $\varphi : \mathbb{D} \longrightarrow X$ holomorphic

$\Longrightarrow f \circ \varphi$ is subharmonic

We say f is plurisubharmonic

Properties:

- f is C^2:

Plurisubharmonic $\Longleftrightarrow i\partial\bar{\partial} f \geqslant 0$

- f is plurisubharmonic

$\Longrightarrow f$ is subharmonic w.r.t any Kähler metric

Definitions:

Let (X, ω) be compact Kähler:

ω-sh: $SH(\omega)$	**ω-Psh**: $PSH(\omega)$
$u \in SH(\omega)$:	$v \in PSH(\omega)$
• u = Locally smooth + subharmonic	• v = Locally smooth + Plurisubharmonic
• $(\omega + i\partial\bar{\partial} u) \wedge \omega^{n-1} \geq 0$ in the sense of distributions	• $\omega + i\partial\bar{\partial} v \geq 0$ in the sense of currents.

$\Longrightarrow$

$\iff \omega^n + i\partial\bar{\partial} u \wedge \omega^{n-1} \geqslant 0$

$\boxed{PSH(\omega) \subseteq SH(\omega)}$

$\iff$

$\Delta_\omega u \geqslant -n$

$\boxed{\begin{array}{l} \longrightarrow \text{Subsets of } PSH(\omega) \text{ have strong compactness and integrability properties.} \\ \forall r \geqslant 1: \quad \bullet\, PSH(\omega) \subset L^r(X) \\ \bullet\, PSH_A(\omega) := PSH(\omega) \cap \{-A \leqslant \sup u \leqslant 0\} \\ \quad \text{is compact in } L^r \end{array}}$

Green's function: (X,ω) Kähler.

G_x is the unique ω-sh function s.t

$$\frac{(\omega + i\partial\bar\partial G_x)\wedge\omega^{n-1}}{V_\omega} = \delta_x$$

and $\int_X G_x\,\omega^n = 0$

Stokes Thm: $u \in SH(\omega)$:

$$u(x) = \int_X u\,\delta_x = \frac{1}{V_\omega}\int u\,(\omega + i\partial\bar\partial G_x)\wedge\omega^{n-1}$$

$$= \frac{1}{V_\omega}\int G_x\,(\omega + i\partial\bar\partial u)\wedge\omega^{n-1}$$

In particular $G_x(y) = G_y(x)$

- $G_x \in C^\infty(X - \{x\})$
$$\lim_{y \to x} G_x(y) = -\infty$$

Green's formula:

$$u(x) - \bar{u} = \frac{-1}{V_\omega} \int_X du \wedge d^c G_x \wedge \omega^{n-1}$$

$$= \frac{-1}{n V_\omega} \int_X \langle \nabla u, \nabla G_x \rangle_\omega \, \omega^n$$

General comparaison

Principle:

Proposition: Fix $t > 0$, $p > 1$

and $0 \leq f \in L^p(\omega^n)$. Let

v (resp. φ) be the unique bounded ω-sh (resp. ω-psh) function s.t :

- $(\omega + i\partial\bar{\partial} v) \wedge \omega^{n-1} = e^{tv} f \omega^n$

- $(\omega + i\partial\bar{\partial}\varphi)^n = e^{nt\varphi} f^n \omega^n$

Then : $\varphi \leq v$

Proof :

- Maximum principle

$$\Rightarrow \quad v(x) = \sup \left\{ u(x),\ u \text{ is a subsolution} \right\}$$

- $$(\omega + i\partial\bar\partial\varphi) \wedge (\omega + i\partial\bar\partial(0))^{n-1}$$

$$\overset{\text{AM-GM}}{\geq} \sqrt[n]{e^{nt\varphi} f^n \cdot 1 \cdot 1 \cdots 1}\ \omega^n$$

$$= e^{t\varphi} f\, \omega^n$$

$\Rightarrow \varphi$ is a subsolution

$$\Rightarrow \quad \varphi \leq v$$

$\square$

This allows to compare

w-sh solutions to the Laplace equation with w-psh solutions to an auxiliary M-A equation.

Uniform estimates for M-A equations:

Theorem [DuGG23, Thm A]:

a X compact Kähler

• X compact Kähler

• ω semi-positive, $V = \int_X \omega^n > 0$

• ν and $\mu = f\nu$ probability measures on X such that

(+) $\|f\|_{L^{p'}(d\nu)} \leq B'$

And

(∗) $\exists\, \alpha > 0$ and $C_\alpha > 0$ s.t

$\forall\, \psi \in PSH(\omega)$.

$C_\alpha \; \alpha(\psi - \sup \psi)$

$$\underbrace{\int_X e^{-\alpha(\varphi - \sup_X \varphi)} \, d\nu \leq C_\alpha}_{\text{integrability condition}}$$

Then $\Longrightarrow$ the unique solution

$\varphi \in PSH(\omega)$, $\sup_X \varphi = 0$:

$$V^{-1}(\omega + i\partial\bar{\partial}\varphi)^n = \mu$$

satisfies $-M \leq \varphi \leq 0$

$$M = M(p, B, \alpha, C_\alpha)$$

Therefore, if we can establish (*) uniformly on $\mathcal{K}(\mathcal{X}, p, A, B)$, we can apply this theorem for $\nu_t = dV_{X_t}$.

Thm: $\exists\, \alpha = \alpha(n, p, A, B)$ and $C = C(\alpha, n, p, A, B)$ s.t $\forall\, \omega \in \mathcal{K}(\mathcal{X}, p, A, B)$ and $\forall\, \varphi_t \in PSH(X_t, \omega_t)$, (*) is satisfied

Proof:

$\square$

Implication for solutions to the Laplace equation:

to the Laplace Equation

Lemma: Fix $a>0$ and let v be a quasi-subharmonic function on X s.t. $\Delta_\omega v \geq -a$ and $\int_X v\,\omega^n = 0$. Then

$$\sup_X v \leq C\left[a + \frac{1}{V_\omega}\int_X |v|\,\omega^n\right],$$

where $C = C(n, p, A, B)$

Prop: Let u be a continuous function s.t

$$\int_X u\,\omega^n = 0 \quad \text{and} \quad |\Delta_\omega u| \leqslant 1$$

Then

$$\|u\|_{L^\infty} \leqslant C = C(n, p, A, B)$$

$\Longrightarrow$ Main theorem:

Theorem: Fix

$0 < r < \frac{n}{n-1}$, $0 < s < \frac{2n}{2n-1}$

$\forall t \in \mathbb{D}^*$, $x \in X_t$ and

$\omega \in \mathcal{K}(\mathcal{X}, p, A, B)$:

① $\sup_{X_t} G_x^{\omega_t} \leq C_0(n, p, A, B)$

② $\frac{1}{V} \int |G_x^{\omega_t}|^r \omega_t^n \leq C_1(n, p, r, A, B)$

$$V_{w_t} \int_{X_t}$$

3 $$\frac{1}{V_{w_t}} \int_{X_t} \left| \nabla G_x^{w_t} \right|_{w_t}^{s} w_t^n \leqslant C_2(n,p,s,A,B)$$

Proof:

1 Let $h = -\mathbb{1}_{\{G_x \leq 0\}}$

$$+ \int \frac{\omega^n}{V_\omega}$$

$$\{G_x \leq 0\}$$

$$\Rightarrow \quad |h| \leq 1 \quad \text{and}$$

$$\int_X h\,\omega^n = 0$$

$$\Rightarrow \quad \text{if } \Delta_\omega v = h, \int v\,\omega^n = 0$$

then, by Prop:

$$\|v\|_{L^\infty(X)} \leq C$$

$$\Rightarrow C \geq v(x) = \frac{1}{V} \int v(\omega + dd^c G_x)$$

$$\frac{}{V_\omega}\int_X \qquad \wedge\omega$$

$$= \frac{1}{V_\omega}\int_X G_x \; dd^c v \wedge \omega^{n-1}$$

$$= \frac{1}{nV_\omega}\int_X G_x \; \Delta_\omega v \cdot \omega^n$$

$$= \frac{1}{nV_\omega}\int_X G_x \cdot h \; \omega^n$$

$$\boxed{\int_X G_x\, \omega^n = 0}$$

$$= \frac{1}{n}\int \left(-G_x\right)\frac{\omega^n}{V_\omega}$$

$\{G_x \leq 0\}$

$$\Rightarrow \int_X |G_x| \frac{\omega^n}{V_\omega} = 2 \int_{\{G_x \leq 0\}} (-G_x) \frac{\omega^n}{V_\omega}$$

$$\leq 2n \cdot C$$

$\Rightarrow$ by Lemma :

$$\boxed{\sup_X G_x \leq C_0}$$

We already proved

(2) We already proved

(2) for $n = 1$. We will prove (2)

for $r < 1 + \frac{1}{n}$,

then using an induction argument, we prove (2)

for $r < 1 + \frac{1}{n} + \frac{1}{n^2} + \cdots + \frac{1}{n^k}$

Hence for $r < \frac{n}{n-1}$.

Now, for $r < 1 + \frac{1}{\ }$,

Notice that :

$$|G_x|^r = \Big| \underbrace{G_x - C_0 - 1}_{:= \tilde{G}_x} + (C_0+1) \Big|^r$$

$$\leqslant \left(-\tilde{G}_x + |C_0+1| \right)^r$$

$$\leqslant (-\tilde{G}_x)^r (1+|C_0+1|)^r$$

$-\tilde{G}_x \geqslant 1$, Hence it is

sufficient to prove 2 for

$-G_x$ for $1 < r < 1 + \frac{1}{n}$

we put $r = 1 + \beta$ for

$0 < \beta < \frac{1}{n}$.

Let u be the ω-sh solution

of $\frac{1}{V_\omega}(\omega + dd^c u) \wedge \omega^{n-1}$

$$= \frac{(-\widetilde{G}_x)^\beta \omega^n}{\int_X (-\widetilde{G}_x)^\beta \omega^n}$$

$\int u \omega^n = 0$. If $u \geq -C$

$$\Rightarrow -C \leq u(x)$$

$$= \int_X u \, \frac{(\omega + dd^c \tilde{G}_x) \wedge \omega^{n-1}}{V_\omega}$$

$$= \int_X \tilde{G}_x \, \frac{(\omega + dd^c u) \wedge \omega^{n-1}}{V_\omega}$$

$$= - \frac{\int_X (-\tilde{G}_x)^{1+\beta} \frac{\omega^n}{V_\omega}}{\int_X (-G_x)^{\beta} \frac{\omega^n}{V_\omega}}$$

$$\Rightarrow \boxed{\int_X \left(-\widetilde{G}_x\right)^{1+\beta} \leq C(1+C_0)}$$

Therefore, it's sufficient to prove $\underline{\underline{u \geq -C}}$

Let $\varphi \in PSH(\omega) \cap L^\infty$ be s.t $\sup_X \varphi = 0$ and

$$(\omega + dd^c\varphi)^n \quad \frac{\left(-\widetilde{G}_x\right)^{n\beta} \omega^n}{\quad}$$

$$\frac{1}{V_\omega}\left(\omega + dd^c\varphi\right) = \frac{\qquad}{\int_X \left(-\widetilde{G}_x\right)^{n\beta}\omega^n}$$

- $$\left(\omega + dd^c\varphi\right)\wedge\omega^{n-1} \overset{\text{AM-GM}}{\geq} \frac{\left(-\widetilde{G}_x\right)^{\beta}\omega^n}{\left(\int_X \left(-\widetilde{G}_x\right)^{n\beta}\right)^{\frac{1}{n}}} \geq \frac{\left(-\widetilde{G}_x\right)\omega^n}{C_1}$$

$$\Rightarrow \boxed{\Delta\varphi \geq -n + \frac{n\left(-\widetilde{G}_x\right)}{C_1}}$$

$$\bullet \quad (\omega + dd^c u) \wedge \omega^{n-1} \leq (-\tilde{G}_x)^{\beta} \omega^n$$

$$(-\tilde{G}_x \geq 1)$$

$$\Rightarrow \quad \Delta u \leq -n + n(-\tilde{G}_x)$$

$$\Rightarrow \quad \boxed{\Delta\left(-\frac{u}{c_1}\right) \geq \frac{n}{c_1} - \frac{n(-\tilde{G}_x)}{c_1}}$$

$$\Rightarrow \quad \Delta\left(\varphi - \frac{u}{c_1}\right) \geq -a = -\left(n - \frac{n}{c_1}\right)$$

Let $v := \varphi - \frac{u}{c_1} - \int \varphi \Rightarrow \int v \omega^n = 0$

and $\Delta v \geqslant -a \xrightarrow{\text{Lemma}} v \leqslant C'$

$\Rightarrow u \geqslant C + C'\varphi$

bounding φ: (Theorem 1.9) ($L^{p'}$ norm)

$$\int_X (-\widetilde{G}_x)^{n\beta p'} f_\omega^{p'} \, dV_x \overset{\text{Hölder}}{\leqslant} \left(\int f_\omega^{p}\right)^{\frac{p'-1}{p-1}} \cdot \left(\int (-\widetilde{G}_x)^{n\beta p' s'}\right)^{\frac{1}{s}} \leqslant C$$

(3) $\int |\nabla G_x|^s \omega^n$

$= \int_X |\nabla \widetilde{G}_x|^s \omega^n$

$= \int \frac{|\nabla \widetilde{G}_x|^s}{\cdot} \cdots |\widetilde{G}|^\alpha \omega^n$

$$\int_X \frac{\cdots}{|\tilde G_x|^\alpha} \cdot |G_x| \cdots \omega$$

Hölder

$$\leq \left(\int_X \frac{|\nabla \tilde G_x|^2}{|\tilde G_x|^{\frac{2\alpha}{s}}} \right)^{\frac{s}{2}} \cdot \left(\int_X |\tilde G_x|^{\frac{2\alpha}{2-s}} \right)^{\frac{2-s}{2}}$$

setting $2\alpha = s(1+\beta)$

and $r = \frac{s}{2-s}(1+\beta)$

$$\leq \left(\int \frac{|\nabla \tilde G_x|^2}{|\tilde G_x|^{1+\beta}} \right)^{\frac{s}{2}} \cdot$$

$$\left(\int_X |\tilde{G}_x|^{\dots}\right)$$

$$\underbrace{\left(\int_X |\tilde{G}_x|^{r}\right)^{\frac{q-s}{q}}}_{Q}$$

Lemma: for $\beta > 0$:

$$\frac{1}{V_\omega}\int_X \frac{dG_x \wedge d^cG_x \wedge \omega^{n-1}}{\left(-\tilde{G}_x\right)^{1+\beta}} \leq \frac{1}{\beta}$$

Proof: $u(y) = \left(-\tilde{G}_x(y)\right)^{-\beta}$

$u \in [0,1]$, $u(x) = 0$

$\beta\, dG_x$

$$du = \frac{\quad}{(-\tilde{G}_x)^{1+\beta}}$$

by Green's formula:

$$u(x) - \bar{u} = -\frac{1}{V_\omega}\int du \wedge d^c G_x \wedge \omega^{n-1}$$

$$0 \geq -\bar{u} = \frac{-\beta}{V_\omega}\int_X \frac{dG_x \wedge d^c G_x \wedge \omega^{n-1}}{(-\tilde{G}_x)^{\beta+1}}$$

$\square$

Sobolev estimates:

Theorem: Fix $1 < r < \frac{2n}{n-1}$,

$t \in \mathbb{D}^*$, $\omega \in \mathcal{K}(X, p, A, B)$

① $\forall u \in H^1(X_t)$,

$$\left(\frac{1}{V_{\omega_t}} \int_{X_t} |u-\bar{u}|^{2r} \omega_t^n\right)^{\frac{1}{r}}$$

$$\leqslant C_1 \frac{1}{V_{\omega_t}} \int_{X_t} |\nabla u|^2_{\omega_t} \omega_t^n$$

② If $\Omega \subset X_t$ is a domain

and $u \in H^1(\Omega)$ is compactly

supported, then

$$\left(\frac{1}{V_{\omega_t}}\int_{X_t}|u|^{2r}\,\omega_t^{n}\right)^{\frac{1}{r}}$$

$$\leq C_2\left[1+\frac{V_{\omega_t}(\Omega)}{V_{\omega_t}(X_t\setminus\Omega)}\right]\cdot$$

$$\frac{1}{V_{\omega_t}}\int_{\Omega}|\nabla u|^2_{\omega_t}\,\omega_t^{n}.$$

Proof:

(1)

$$|u(x) - \bar{u}| = \left| \frac{1}{V_\omega} \int_X du \wedge d^c \tilde{G}_x \wedge \omega^{n-1} \right|$$

Hölder

$$\leq \left(\frac{1}{V_\omega} \int_X \frac{d\tilde{G}_x \wedge d^c \tilde{G}_x \wedge \omega^{n-1}}{(-\tilde{G}_x)^{\beta+1}} \right)^{\frac{1}{2}}$$

$$\cdot \left(\frac{1}{V_\omega} \int_X (-\tilde{G}_x)^{\beta+1} |\nabla u|^2_{\omega^n} \right)^{\frac{1}{2}}$$

$$\leqslant \frac{1}{\beta^{\frac{1}{2}}} \cdot \Bigg(\underbrace{\frac{1}{V_\omega} \int_X (-\tilde{G}_x)^{\beta+1} |\nabla u|^2 \omega^n}_{\varphi} \Bigg)^2$$

$$\Rightarrow \left(\int |u-\bar{u}|^{2r} \right)^{\frac{1}{r}} \leqslant \frac{1}{\beta} \| \varphi \|_{L^r}$$

by Minkowski's inequality:

$$\|\varphi\|_{L^r} \leqslant \frac{1}{V_\omega} \int_X \left(\int_X (-\tilde{G}_x)^{r(\beta+1)} \omega^n \right)^{\frac{1}{r}} |Du|^2 \omega^n$$

2

$$V^{\frac{1}{r}}$$

$$\leq C_1 \frac{V_{\omega}}{V_{\omega}} \int_X |\nabla u|_{\omega} \, \omega^n$$

Hence 1 is proved

2 for $x \in \Omega^c$, we have :

$$0 = u(x) = \bar{u} - \frac{1}{V_{\omega}} \int du \wedge d^c G_x \wedge \omega^{n-1}$$

integrating over Ω^c

$$\Longrightarrow$$

$$\bar{u} = \frac{1}{V_\omega(\Omega^c)} \int_{\Omega^c} \frac{1}{V_\omega} \int_X du \wedge d^c \tilde{G}_x \wedge \omega^{n-1}$$

$$\leq \left(\frac{C}{V_\omega(\Omega^c)} \int_X |\nabla u|^2 \omega^n \right)^{\frac{1}{2}}$$

$$\circledast$$

$$\Longrightarrow |u(x)|^2 \leq |\bar{u}|^2 + \frac{1}{\beta V_\omega} \int (-\tilde{G}_x)^{1+\beta} |\nabla u|^2_{\omega^n}$$

we use $(*)$ and we proceed as before.

$\square$