Lemma: $(X,\omega) = (X_t,\omega_t)$, $\omega \in \mathcal{K}(p, A, B)$

- v quasi-sh
- $\int_X v\,\omega^n = 0$
- $\Delta_\omega v \geq -a$

$$\implies \sup_X v \leq C\left[a + \frac{1}{V_\omega}\int_X |v|\,\omega^n\right]$$

Proposition:

- u continuous
- $\int_X u\,\omega^n = 0$
- $|\Delta_\omega u| \leq 1$

$$\Longrightarrow \quad \|u\|_{L^\infty(X)} \leq C$$

Proof of Lemma:

- Assumptions + statement are hom. of deg $1 \Rightarrow a = n$

$$\Delta \ldots \; u \; \Longleftrightarrow (\omega + dd^c v) \wedge \omega^{n-1}$$

$\Delta_\omega v \geq -n \iff (\omega + dd^c v)^n \geq 0$

- Regularizing v, we can suppose v is smooth.

- $v_+ = \widetilde{\max}(v, 0)$

$\widetilde{\max}$: convex regularized maximum s.t

$$0 \leq \widetilde{\max} \leq \max + 1$$

$\Rightarrow$ • $M = \int v_+ \cdot \frac{\omega^n}{V_\omega} \leq 1 + \frac{1}{2} \int |v| \omega^n$

- $\sup_X v \leq \sup_X v_+$

$\Rightarrow$ it's sufficient to prove:

$$\boxed{v_+ \leq C_0 + C'M}$$

We consider $\varphi \in PSH(X,\omega)$ smooth:

- $(\omega + dd^c \varphi) = \dfrac{1+v_+}{1+M}\,\omega^n$

- $\sup \varphi = -1$

$\bullet \ \sup_x$

$$\Leftrightarrow \left\{ -\varphi = e^{-f}, \ \sup_x f = 0 \right.$$

Uniform integrability $\Longrightarrow$

$$\exists \delta > 0 \text{ s.t.: } \int (-\varphi)^{\delta} \, dV_x = \int e^{-\delta f} \, dV_x \leq C_\delta$$

__Step 1:__

$$\bullet \ v_+ \leq v_+ + 1 \leq \varepsilon (-\varphi)^{\alpha}$$

for $\begin{cases} \alpha = \dfrac{n}{n+1} \\[2ex] \dfrac{\varepsilon^{n+1}\alpha^n}{(1+\alpha\varepsilon)^n} = 1+M \\[2ex] \left(\text{in particular, } \varepsilon \leqslant c_n(1+M)\right) \end{cases}$

<u>Step 2</u>: L^r-bound and M-A estimate

$$\frac{(\omega+dd^c\varphi)}{V_\omega} = \underbrace{\frac{1+v_+}{1+M} f}_{g}\, dV_x$$

$$g \leq \frac{\varepsilon(-\varphi)^{\alpha}}{1+\mu} f \leq c_n (-\varphi)^{\alpha} \cdot f$$

$$\Longrightarrow \|g\|^r_{L^r(dV_x)} \leq c_n \int (-\varphi)^{\alpha r} f^r \, dV_x$$

$$\leq C_n \underbrace{\left(\int f^p\right)^{\frac{r}{p}}}_{\leq B} \underbrace{\left(\int (-\varphi)^{\frac{\alpha r p}{p-r}}\right)^{\frac{p-r}{p}}}_{\leq C_\alpha}$$

M-A estimate $\Longrightarrow$ φ uniformly bounded

$$\Longrightarrow v_+ \leqslant \varepsilon(-\varphi)^\alpha$$

$$\leqslant C_n(M+1)\cdot C_0$$

and the proof is finished

<u>Proof of Step 1:</u>

$$H := v_+ + 1 - \varepsilon(-\varphi)^\alpha$$

we want to prove $H \leqslant 0$

notice that

$$-dd^c(-\varphi)^\alpha = \alpha(\alpha+1)(-\varphi)^{\alpha-2} d\varphi \wedge d^c\varphi$$

$$+ \alpha(-\varphi)^{\alpha-1} dd^c \varphi$$

$$\Rightarrow \Delta_\omega\left(-\varepsilon(-\varphi)^\alpha\right) \geqslant \varepsilon\alpha(-\varphi)^{\alpha-1}\Delta_\omega\varphi$$

AM-GM

$$\geqslant n\alpha\varepsilon(-\varphi)^{\alpha-1}\left[\left[\frac{1+v_+}{1+M}\right]^{\frac{1}{n}} - 1\right]$$

$$\Rightarrow \Delta_\omega H \geqslant -n + n\alpha\varepsilon(-\varphi)^{\alpha-1} \cdot \left[\left[\frac{1+v_+}{1+M}\right]^{\frac{1}{n}} - 1\right]$$

but at x_0 where H reaches its maximum, we have:

$$\Delta_\omega H \leq 0$$

$$\Longrightarrow$$

$$(1+\alpha\varepsilon)(-\varphi)^{1-\alpha} \geq (-\varphi)^{1-\alpha} + \alpha\varepsilon$$

$$\geq \alpha\varepsilon\left[\frac{1+v_+}{1+M}\right]^{\frac{1}{n}}$$

$$\Rightarrow \quad \varepsilon(-\varphi)^{n(1-\alpha)} \geq \frac{\alpha^n\varepsilon^{n+1}}{(1+\alpha\varepsilon)^n}\cdot\frac{1+v_+}{1+M}$$

$= 1 + \delta_+$

$\Rightarrow \quad H \leq 0$

$\square$

Proof of proposition:

Using Lemma, it's sufficient to prove that

if u cont. $\int u w^n = 0$

$|\Delta_w u| \leq S(n, p, A, B)$

$$\Rightarrow M := \frac{1}{V_\omega}\int |\varphi|\omega \leq C \text{ uniformly.}$$

We assume $M \geq 1$

(else we're done).

We set

$$v = \frac{u}{M} = \varepsilon u$$

$$0 < \varepsilon := \frac{1}{M} \leq 1$$

therefore $\displaystyle\frac{1}{V_\omega}\int |v|\,\omega^n = 1$

Lemma $\Longrightarrow \quad \|v\|_{L^\infty} < C$

We would like to Prove that

$$\|v\|_{L^\infty} \leqslant C \cdot \varepsilon + \frac{1}{2}$$

$$\Longrightarrow 1 = \frac{1}{V_w}\int |v| \omega^n \leqslant \frac{1}{V_w}\int \|v\|_{L^\infty} \omega^n$$

$$\leqslant C \cdot \varepsilon + \frac{1}{2}$$

$$\Rightarrow \quad \frac{1}{2C} \leqslant \varepsilon = \frac{1}{M}$$

$$\boxed{\Rightarrow M \leq 2C}$$

- v is an $\varepsilon\omega$-sh which solves

$$(\varepsilon\omega + dd^c v) \wedge (\varepsilon\omega)^{n-1}$$

$$= \varepsilon^n (\omega + dd^c u) \wedge \omega^{n-1}$$

$$= \varepsilon^n \big(1 + \underbrace{\Delta_\omega u}_{:= H}\big)\, \omega^n$$

$$= \varepsilon^n (1 + H)\, \omega^n$$

$$^{tv} \qquad ^{-tv} \qquad (\varepsilon\omega)^n$$

$$= e \quad e \quad (1+H) \quad (\varepsilon\omega)$$

• Let φ be the $\varepsilon\omega$-psh solution to

$$(\varepsilon\omega + dd^c\varphi)^n = e^{nt\varphi} e^{-nt v} (1+H)^n (\varepsilon\omega)^n$$

$$\Longrightarrow \varphi \leq v$$

thus using a similar argument for $-v$, it's sufficient to prove $\varphi \geq -C\varepsilon - \frac{1}{2}$

- $|H| = |\Delta_\omega u| \leq \delta \leq 1$
- $\varphi \leq v \Rightarrow \varphi - v \leq 0$

$$\Rightarrow (\varepsilon\omega + dd^c\varphi)^n \leq 2^n(\varepsilon\omega)^n$$

- setting $\psi = \frac{\varphi}{\varepsilon}$

$$\Rightarrow (\omega + dd^c\psi) \leq 2^n \omega^n$$

$$\xRightarrow{\text{uniform integrability + MA-Estimates}}$$

for $\tilde{\psi} := \psi - \sup_X \psi$, we get

$$\|\tilde{\psi}\|_{L^\infty} \leq C_0$$

$$\Rightarrow \boxed{\left(\varphi - \sup_X \varphi\right) \geq -C_0 \varepsilon}$$

1

- By integration,

$$1 = \int_X \frac{\omega^n}{V_\omega} = \int_X \frac{(\omega + dd^c \psi)^n}{V_\omega}$$

$$= \int_X e^{nt\varepsilon\psi} e^{-nt\varepsilon v} (1+H)^n \frac{\omega^n}{V_\omega}$$

$$= e^{nt\varepsilon \sup \psi} \int_X e^{nt\varepsilon \psi} e^{-nt v} (1+H)^n \frac{\omega^n}{V_\omega}$$

$$\leqslant e^{nt\varepsilon \sup\psi} (1+\delta)^n \int_X e^{-ntv} \frac{\omega^n}{V_\omega}$$

using $e^x \leqslant 1 + x + x^2$ for $|x| \leqslant 1$

since $\|v\| \leqslant C$, for

$t \leqslant n^{-1} C^{-1} \Rightarrow |ntv| \leqslant 1$

$\Rightarrow$

$$\int e^{-ntv} \frac{\omega^n}{V_\omega} \leqslant 1 - nt \int v \omega^n$$

$$\int \quad v_w \geq \int$$

$$+n^2t^2\int v^2 w^n$$

$$\left(\int v w^n = 0\right)$$

$$\leq 1+n^2t^2\int v^2 w^n$$

$$\leq 1+n^2t^2 C^2$$

$$\Rightarrow e^{nt\varepsilon \sup \psi} \geq \frac{1}{(1+\delta)^n(1+n^2t^2C^2)}$$

$$\Rightarrow nt\varepsilon \sup \psi \geq -n\log(1+\delta) - \log(1+n^2t^2C^2)$$

$$\geq -n\delta - n^2 t^2 C^2$$

$$\Rightarrow \quad \mathbb{E} \sup \Psi \geq -\frac{\delta}{t} - n t C^2$$

choosing $t = \sqrt{\delta} = \frac{1}{2(1+nC^2)}$

$$\Rightarrow \quad \mathbb{E} \sup \Psi \geq -\sqrt{\delta}\,(1+nC^2)$$

$$= -\frac{1}{2}$$

$$\Rightarrow \quad \boxed{\sup \varphi \geq -\frac{1}{2}}$$

②

① + ② $\Longrightarrow$ $\varphi \gtrsim -C_0\, \varepsilon^{-\frac{1}{2}}$

$\blacksquare$