Applications:

Theorem: Fix $0 < \delta < 1$ and $\omega \in \mathcal{K}(X, p, A, B)$.

$\exists C(n, p, \delta, A, B) > 0$ s.t $\forall t \in \mathbb{D}^*$

$x \in X_t$, $r > 0$:

① $\cdot\ \operatorname{diam}(X_t, \omega_t) \leq C$

② $\cdot\ \dfrac{\operatorname{Vol}_{\omega_t}(B_{\omega_t}(x, r))}{V_{\omega_t}} \geq C \min\{1, r^{2n+\delta}\}$

$\Longrightarrow \{(X_t, d_{\omega_t}), \omega \in \mathcal{K}(X, p, A, B), t \in \mathbb{D}^*\}$ is pre-compact in the G-H topology

Proof:

① Let $x_0, y_0 \in X$ s.t $d(x_0, y_0) = \mathrm{diam}(X, \omega)$

$$\rho : X \longrightarrow \mathbb{R}_+$$
$$x \longrightarrow d_\omega(x, x_0)$$

$\Rightarrow \rho$ is 1-Lipschitz and $\rho(x_0) = 0$

Green's formula $\Longrightarrow$

$$0 = \rho(x_0) = \frac{1}{V_\omega}\int_X \rho\, \omega^n - \frac{1}{nV_\omega}\int_X \langle \nabla\rho, \nabla G_{x_0} \rangle \omega^n$$

$$\Rightarrow \frac{1}{V_\omega}\int_X \rho\, \omega^n = \frac{1}{nV_\omega}\int_X \langle \nabla\rho, \nabla G_{x_0} \rangle \omega^n$$

$$\leq \frac{1}{nV_\omega}\int_X |\nabla\rho| \cdot |\nabla G_{x_0}| \,\omega^n$$

$$\leq \frac{1}{nV_\omega}\int_X |\nabla G_x| \,\omega^n \leq C$$

$\Longrightarrow$ by Green's formula:

$$\operatorname{diam}(X,\omega) = d(x_0, y_0)$$

$$= \rho(y_0) = \frac{1}{V_\omega}\int \rho\,\omega^n - \frac{1}{V_\omega}\int \langle \nabla\rho, \nabla G_{y_0}\rangle \omega^n$$

$$\leq C + C = 2C$$

2. Fix $x \in X$, $r \in [0,1]$.

$$\rho : X \longrightarrow \mathbb{R}^+$$

$$\rho : X \longrightarrow \mathbb{R}$$
$$y \longrightarrow d(x,y)$$

$\Longrightarrow$ by (1), ρ is uniformly bounded

Let φ be a cutoff function s.t.:

$$\begin{cases} \bullet\ \mathrm{Supp}(\varphi) \subset B(x,r) \\ \bullet\ \varphi \equiv 1 \text{ on } \overline{B\left(x, \frac{r}{2}\right)} \\ \bullet\ \sup_x |D\varphi|_\omega \leqslant \frac{C}{r} \end{cases}$$

$\Rightarrow \rho \circ \varphi$ is C-Lipschitz

Let $s \in \left(1, \frac{2n}{2n-1}\right)$ and

$$s^* = \frac{s}{s-1} \in (2n, \infty)$$

For $y \in \overline{B(x,r)}^c$, By Green:

$$\int_X \rho\varphi \, \omega^n = \frac{1}{n}\int \langle \nabla(\rho\varphi), \nabla G_y \rangle \omega^n$$

Hölder

$$\leq \left(\int |\nabla G_y|^s\right)^{\frac{1}{s}} \frac{1}{n} \left(\int |\nabla \rho\varphi|^{s^*}\right)^{\frac{1}{s^*}}$$

$$\leq C \cdot V_\omega^{\frac{1}{s}} \, Vol_\omega(B(x,r))^{\frac{1}{s^*}}$$

$$\Longrightarrow \text{For } z \in \partial B\left(x, \frac{r}{2}\right),$$

By Green:

$$\frac{r}{2} = \rho\varphi(z) = \frac{1}{V_\omega}\int \rho\varphi\,\omega^n$$

$$+ \frac{1}{V_\omega}\int d(\rho\varphi) \wedge d^c G_z \wedge \omega^{n-1}$$

$$\leq 2C\, V_\omega^{-1+\frac{1}{s}} \, Vol_\omega(B(x,r))^{\frac{1}{s^*}}$$

$$= 2C\left(\frac{Vol(B(x,r))}{V_\omega}\right)^{\frac{1}{s^*}}$$

$\Rightarrow$ since $s^* \in (2n, \infty)$,

we get 2

$\square$

Corollaries:

CSCK metrics:

- (X,β) Kähler variety with klt singularities

- $\pi : \mathcal{X} \longrightarrow \mathbb{D}$ a $\mathbb{Q}$-Gorenstein smoothing

- $\beta_{\mathcal{X}}|_{X_t} = \beta_t$, $\beta_{\mathcal{X}}|_{X_0} = \beta$

by [Pan, Tô, Trusiani], if the Mabuchi functional

M is coercive $\Longrightarrow M$ is

M_β coercive $\Rightarrow M_{\beta_t}$

coercive $\Longrightarrow \exists! \; \omega_t \in [\beta_t]$

cscK

Corollary 1:

$$\boxed{\operatorname{diam}(X_t, \omega_t) \leqslant D}$$

Proof: $\omega_t = \beta_t + dd^c \varphi_t$ solves

$$\begin{cases} (\beta_t + dd^c \varphi_t)^n = e^{F_t} \beta_t^n \\ \\ \Delta_{\omega_t} F_t = -\bar{s}_t + \operatorname{Tr}_{\omega_t} \operatorname{Ric}(\beta_t) \end{cases}$$

$\Rightarrow$ • $\{\beta_t\}$ are uniformly bounded away from 0 and ∞

• $[\omega_t] = [\beta_t]$

• By [PTT, Theorem 5.3]

$f_t = e^{F_t}$ verify

$$\|f_t\|_{L^p(X_t, \beta_t)} \leq B$$

$\Rightarrow \; \omega \in \mathcal{K}(\mathcal{X}, p, 1, B)$

$\square$

Similar results for KRF and CY

Sobolev estimates:

Theorem: Fix $1 < r < \frac{2n}{n-1}$,

$t \in \mathbb{D}^*$, $\omega \in \mathcal{K}(\mathcal{X}, p, A, B)$

① $\forall u \in H^1(X_t)$,

$$\left(\frac{1}{V_{\omega_t}}\int_{X_t}|u-\bar{u}|^{2r}\omega_t^n\right)^{\frac{1}{r}}$$

$$\leq C_1\frac{1}{V_{\omega_t}}\int_{X_t}|\nabla u|^2_{\omega_t}\omega_t^n$$

② If $\Omega\subset X_t$ is a domain and $u\in H^1(\Omega)$ is compactly supported, then

$$\left(\frac{1}{V}\int|u|^{2r}\omega_t^n\right)^{\frac{1}{r}}$$

$$\leq C_2\left[1+\frac{V_{\omega_t}(\Omega)}{V_{\omega_t}(X_t\setminus\Omega)}\right]\cdot\frac{1}{V_{\omega_t}}\int_\Omega |\nabla u|^2_{\omega_t}\,\omega_t^n.$$

Proof:

①

$$|u(x)-\bar{u}| = \left|\frac{1}{V_\omega}\int_X du \wedge d^c \tilde{G}_x \wedge \omega^{n-1}\right|$$

Hölder

$$\leqslant \left(\frac{1}{V_\omega}\int_X \frac{d\tilde{G}_x \wedge d^c\tilde{G}_x \wedge \omega^{n-1}}{(-\tilde{G}_x)^{\beta+1}}\right)^{\frac{1}{2}}$$

$$\cdot \left(\frac{1}{V_\omega}\int_X (-\tilde{G}_x)^{\beta+1} |\nabla u|^2_{\omega^n}\right)^{\frac{1}{2}}$$

$$\leqslant \frac{1}{\beta^{\frac{1}{2}}} \cdot \underbrace{\left(\frac{1}{V_\omega}\int_X (-\tilde{G}_x)^{\beta+1} |\nabla u|^2_{\omega^n}\right)^{\frac{1}{2}}}_{\varphi}$$

$$\Big(\qquad \Big)^{\frac{1}{n}}$$

$$\Rightarrow \left(\int |u-\bar{u}|^{2r} \right)^{\frac{1}{r}} \leqslant \frac{1}{\beta} \|\varphi\|_{L^r}$$

by Minkowski's inequality:

$$\|\varphi\|_{L^r} \leqslant \frac{1}{V_\omega} \int_X \left(\int_X (-\tilde{G}_x)^{r(\beta+1)} \omega^n \right)^{\frac{1}{r}} |Du|^2 \omega^n$$

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$$\leqslant C_1 \frac{V_\omega^{\frac{1}{r}}}{V_\omega} \int_X |Du|_\omega^2 \omega^n$$

Hence 1 is proved

(2) for $x \in \Omega^c$, we have:

$$0 = u(x) = \bar{u} - \frac{1}{V_\omega}\int du \wedge d^c G_x \wedge \omega^{n-1},$$

integrating over Ω^c

$$\Longrightarrow \bar{u} = \frac{1}{V_\omega(\Omega^c)} \int_{\Omega^c} \frac{1}{V_\omega}\int_X du \wedge d^c G_x \wedge \omega^{n-1}$$

$$\leq \left(\frac{C}{V_\omega(\Omega^c)} \int_X |\nabla u|^2 \omega^n \right)^{\frac{1}{2}}$$

...(*)

$\Longrightarrow$

$$|u(x)| \leq \left(|u| + \frac{\;}{\beta V_0} \right) (\sigma_x) \, \|u\|_{D''}$$

we use $(*)$ and we proceed as before.