Weighted Extremal Kähler metrics on resolutions of singularities

joint paper with Sébastien Boucksom and Mattias Jonsson.

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Antonio Trusiani
Chalmers University of Technology

GdT Complexe CIRGET

Université du Québec à Montréal

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Outline

Introduction

Let X be a compact Kähler manifold, let $T \subset \mathrm{Aut}_r(X)$ be a maximal compact torus, and let $\Omega = (\omega, m_\Omega)$ be a T-equivariant Kähler form where $m_\Omega : X \to \mathfrak{t}^\vee$ is a moment map for ω.

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★ $\mathrm{Aut}_r(X)$ is the identity component of the subgroup of automorphisms acting trivially on the Albanese torus $\mathrm{Alb}(X) = H^0(X, \Omega^1_X)^\vee / H_1(X, \mathbb{Z})$.

- Its Lie algebra consists of all holomorphic vector fields $\xi \in H^0(X, TX)$ such that $\alpha(\xi) = 0$ for any holomorphic 1-form $\alpha \in H^0(X, \Omega^1_X)$.
- if X is projective, then $\mathrm{Aut}_r(X) \simeq \mathrm{Aut}_0(X, L)/\mathbb{C}^*$ for $L \to X$ ample line bundle;
- In general, $\mathrm{Aut}_r(X)$ is still a linear algebraic group and $\mathrm{Aut}_0(X)/\mathrm{Aut}_r(X)$ is a compact complex torus [Fujiki '78].

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Definition (Lahdili '19, Inoue '22)

Let $v, w \in C^\infty(\mathfrak{t}^\vee)$ be two weights such that $v, w > 0$ on $P := m_\Omega(X)$. Then Ω is a (v, w)-extremal metric if it is a $(v, w\, l^{\mathrm{ext}})$-cscK metric, i.e.

$$S_v(\Omega) = w(m_\Omega) l^{\mathrm{ext}}(m_\Omega)$$

where $S_v(\Omega)$ is the *v-weighted scalar curvature* while l^{ext} is the unique affine function on $\mathfrak{t}^\vee$ (*v-weighted extremal function*) determined by the vanishing of the weighted Futaki invariant. [$\rightsquigarrow$ Simon's lectures]

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- $(v, w) = (1, 1) \rightsquigarrow$ classical cscK and extremal metrics.
- $w(\alpha) = n + \langle (\log v)'(\alpha), \alpha \rangle \rightsquigarrow$ v-solitons, i.e. $\mathrm{Ric}^T_v(\Omega) = \Omega$.
  Case $v(\alpha) = e^{2\langle \alpha, \xi \rangle} \rightsquigarrow$ gradient Kähler-Ricci solitons $\mathrm{Ric}(\omega) = \omega - \mathcal{L}_{J\xi}\omega$.
- many others...

Let $\mathcal{H}_\omega := \{\varphi \in \mathcal{C}^\infty(X) \,:\, \omega_\varphi := \omega + dd^c\varphi > 0\}$ be the set of Kähler potentials, and denote by $\mathcal{H}_\omega^T$ those that are T-invariant.

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Theorem (Lahdili '19)

Let $v, w \in \mathcal{C}^\infty(\mathfrak{t}^\vee)$ be two weights such that $v, w > 0$ on $P = m_\Omega(X)$. The operator

$$\mathcal{H}_\omega^T \ni \varphi \to \left(w(m_{\Omega_\varphi})I^{\mathrm{ext}}(m_{\Omega_\varphi}) - S_v(\Omega_\varphi)\right) v(m_{\Omega_\varphi})\omega_\varphi^n$$

admits an Euler-Lagrange functional: the weighted relative Mabuchi functional $\mathrm{M}^{\mathrm{rel}}_{\omega,v,w}$.

$\mathrm{M}^{\mathrm{rel}}_{\omega,v,w} : \mathcal{H}_\omega^T \to \mathbb{R}$ is an Euler-Lagrange functional for the operator $\mathcal{H}_\omega^T \ni \varphi \to \mu_\varphi := \left(w(m_{\Omega_\varphi})I^{\mathrm{ext}}(m_{\Omega_\varphi}) - S_v(\Omega_\varphi)\right) v(m_{\Omega_\varphi})\omega_\varphi^n$ means that

$$\frac{d}{dt}\left(\mathrm{M}^{\mathrm{rel}}_{\omega,v,w}(\varphi + t\,f)\right)_{|t=0} = \int_X f\,\mu_\varphi$$

for any $f \in \mathcal{C}^\infty(X)^T$.

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$\rightsquigarrow \varphi \in \mathcal{H}_\omega^T$ is a critical point of $\mathrm{M}^{\mathrm{rel}}_{\omega,v,w}$ iff Ω_φ is a (v, w)-extremal Kähler metric.

Rmk: the same result holds if $T \subset \mathrm{Aut}_r(X)$ is not maximal.

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Theorem (Chen-Cheng '21, Apostolov-Jubert-Lahdili '23, He '19, Di Nezza-Jubert-Lahdili '24, Han-Liu '24)

Under the aforementioned setting,

- the existence of a (v, w)-extremal Kähler metric in $\{\omega\}$ implies that $\mathrm{M}^{\text{rel}}_{\omega,v,w}$ is coercive on $\mathcal{H}_\omega^T$, i.e. there exist $\delta > 0$, $C > 0$ such that $\mathrm{M}^{\text{rel}}_{\omega,v,w} \geq \delta J_{\omega,T} - C$;

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- if v is further log-concave on P, the converse holds as well.

Theorem

Let (X, ω_X) be a compact Kähler space with log terminal singularities, and suppose given:

- a compact torus $T \subset \mathrm{Aut}_r(X)$ preserving ω_X;
- two smooth positive weights v, w on the moment polytope P;
- a T-equivariant resolution of singularities $\pi : Y \to X$, assumed to be of Fano type;
- a sequence of T-invariant Kähler forms ω_j on Y converging smoothly to $\pi^*\omega_X$ and such that $\omega_j \geq (1 - \varepsilon_j)\pi^*\omega_X$ with $\varepsilon_j \to 0$.

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If the relative weighted Mabuchi energy $\mathrm{M}^{\mathrm{rel}}_{\omega_X,v,w}$ is coercive on $\mathcal{E}^{1,T}_{\omega}$, then so is $\mathrm{M}^{\mathrm{rel}}_{\omega_j,v,w}$ on $\mathcal{H}^T_{\omega_j}$ for all j large enough, with uniform coercivity constants.

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- $\mathcal{E}^{1,T}$ is the space of *T-invariant finite energy potentials*, and it is given by the T-invariant elements in the metric completion of $\mathcal{H}_\omega$ with respect to a metric d_1. Note that $c_\omega J_{\omega,T}(\varphi) \leq d_1(\varphi, 0) \leq C_\omega J_{\omega,T}(\varphi) + A_\omega$ for any $\varphi \in \mathcal{E}^{1,T}_{\omega,0}$.

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Corollary

Assume that X is smooth, T is a maximal torus and v is log-concave. If $\{\omega_X\}$ contains a (v, w)-weighted extremal Kähler metric, then so does $\{\omega_j\}$ for all j large enough.

Fano type assumption

Definition

We say that $\pi : Y \to X$ is of Fano type if there exists a singular metric ϕ on $-K_Y$ such that

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- $\iff$ There exists a q-psh function f (i.e. locally sum of a psh function and a smooth function) such that the measure $\hat{\nu}_Y := e^{-2f}\omega_Y^n$ has finite total mass and $\mathrm{Ric}(\omega_Y) + dd^c f$ is π-semipositive;
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Question: When does a resolution of Fano type exist?

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Question: When does a resolution of Fano type exist? Examples:

- $\dim X = 2$;
- $\dim X = 3$, X with Gorenstein quotient singularities;
- X with isolated singularities locally isomorphic to an affine cone over a Fano manifold.
- any crepant resolution.

The main Corollary is related to (X smooth)

- Arezzo-Pacard '06: blowups of points, X with no nonzero holomorphic vector fields.
- Arezzo-Pacard '09: blowups of a large and suitable collection of points, for cscK metrics.
- Arezzo-Pacard-Singer '11: blowup of points, conditions involving the automorphism group, for extremal metrics.
- Székelyhidi '12: reduced the conditions in APS11 to a stability condition.
- Hallam '23: generalized Szé12 to the weighted case.
- Székelyhidi '15, Dervan-Sektnan '21: relate the existence of extremal metric on the blow-up at a point to K-stability.
- Seyyedali-Székelyhidi '20: blowup of T-invariant submanifolds of codimension larger than 2, extremal metrics.

Moreover, for X singular,

- Székelyhidi '24: independently obtain the existence of cscK metrics on the resolution of a KE klt space with no nontrivial holomorphic vector fields.
- Pan-Tô '24: coercivity of $\mathrm{M}^{\mathrm{rel}}_{\omega_X,v,w}$ on $\mathcal{E}^{1,T}_{\omega}$ implies existence of singular (v,w)-weighted extremal Kähler metrics when T is maximal, v is log-concave and the variety admits a T-equivariant resolution of singularities of Fano type.

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One could wonder what happens when the morphisms is not T-equivariant wrt a *maximal* compact torus $T \subset \mathrm{Aut}_r(X)$. For instance when π is T-equivariant wrt to a maximal torus in $\mathrm{Aut}_r(Y)$ which is not maximal in $\mathrm{Aut}_r(X)$.

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By [Apostolov-Jubert-Lahdili] $\mathrm{M}^{\mathrm{rel}}_{\omega_X,v,w}$ should then be assumed to be coercive modulo (the identity component) of the centralizer $\mathrm{Aut}^T_r(X)$ of T in $\mathrm{Aut}_r(X)$, and the conclusion should be (under appropriate assumptions) that $\mathrm{M}^{\mathrm{rel}}_{\omega_j,v,w}$ is coercive modulo $\mathrm{Aut}^T_r(Y) = T_{\mathbb{C}}$.

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The main disadvantage of the approach presented is that it does not cover this case. However, the problem is obstructed in such case [Dervan-Sektnan '21, Hallam '23].

Strong topology of ω-psh functions

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- When $\omega > 0$ then $\mathrm{PSH}(X,\omega) \supset \mathcal{H}_\omega$.

Lemma

The Monge-Ampère operator $\mathcal{C}^\infty(X) \ni \varphi \longrightarrow \mathrm{MA}_\omega(\varphi) := \omega_\varphi^n$ admits a Euler-Lagrange functional $\mathrm{E}_\omega : \mathcal{C}^\infty(X) \to \mathbb{R}$, normalized by $\mathrm{E}_\omega(0) = 0$, called *Monge-Ampère energy wrt* ω.

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    Proof Let $v_j := \left(\sup_{k\geq j} u_k\right)^*$. Then $\mathrm{E}_\omega(u_j) \leq \mathrm{E}_\omega(v_j) \searrow \mathrm{E}_\omega(u)$.

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Examples:

- Assume $n = 1$, then $u \in \mathcal{E}^1_\omega$ iff $\int_X u\,\omega_u > -\infty$. This is equivalent to $\int_X du \wedge d^c u < +\infty$, i.e. $\mathcal{E}^1_\omega = \mathrm{PSH}(X,\omega) \cap W^{1,2}(X)$.
  In higher dimension, you get $\mathrm{E}_\omega(u) \leq \sup_X u - \int_X du \wedge d^c u \wedge \omega^{n-1}$. Hence functions in $\mathcal{E}^1_\omega$ have gradient in $L^2(X)$;
- Let $\varphi \in \mathrm{PSH}(X,\omega), \varphi \leq -1$. Then $-(-\varphi)^\varepsilon \in \mathcal{E}^1_\omega$ if $\varepsilon < \frac{1}{n+1}$.
- If $u \in \mathcal{E}^1_\omega, v \in \mathrm{PSH}(X,\omega), v \geq u$ then $v \in \mathcal{E}^1_\omega$. In particular $\max(u,-k) \in \mathcal{E}^1_\omega$ for any $u \in \mathrm{PSH}(X,\omega)$, and $u \in \mathcal{E}^1_\omega$ if and only if $\mathrm{E}_\omega(\max(u,-k)) > -C$ uniformly.

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Theorem (Berman-Boucksom-Guedj-Zeriahi '13)

- The mixed Monge-Ampère operator admits a unique strongly continuous extension to $\mathcal{E}^1_\omega$. In particular the Monge-Ampère operator is strongly continuous.
- For all $u_0, u_1, \dots, u_n \in \mathcal{E}^1_\omega$, $u_0 \in L^1(\omega_{u_1} \wedge \omega_{u_n})$ and

  $$(u_0, u_1, \dots, u_n) \longrightarrow \int_X u_0\, \omega_{u_1} \wedge \cdots \wedge \omega_{u_n}$$

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Definition (Darvas '15)

The *Darvas metric* on $\mathcal{E}^1_\omega$ is defined as

$$d_1(u,v) := \mathrm{E}_\omega(u) + \mathrm{E}_\omega(v) - 2\,\mathrm{E}_\omega\left(P_\omega(u,v)\right)$$

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- One immediately have $|\mathrm{E}_\omega(u) - \mathrm{E}_\omega(v)| \leq d_1(u,v)$, and it is possible to prove that

$$d_1(u,v) \approx \mathrm{I}_1(u,v) := \int_X |u-v| \left(\mathrm{MA}_\omega(u) + \mathrm{MA}_\omega(v)\right).$$

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- $(\mathcal{E}^1_\omega, d_1)$ is a geodesic metric space. Indeed any two points in $\mathcal{E}^1_\omega$ can be joined by a unique *psh geodesic* $(u_t)_{t\in[0,1]}$, which is a constant speed geodesic for d_1, i.e. $d_1(u_t,u_s) = |t-s|d_1(u_0,u_1)$.

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Action by automorphisms

As the neutral component $\mathrm{Aut}_0(X)$ acts trivially on cohomology, for each $g \in \mathrm{Aut}_0(X)$ there exists a unique $\tau_g \in C^\infty(X)$ such that

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Finally, assume $\omega > 0$. Since $\mathrm{Aut}_0(X)$ acts by isometries, we can assume $u = 0$. By closedness of $\{g \in \mathrm{Aut}_0(X) \,:\, d_1(\tau_g, 0) \leq C\}$, it is enough to show that any sequence g_j such that $d_1(\tau_{g_j}, 0) \leq C$ admits a convergent subsequence in $\mathrm{Aut}(X)$.

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This extends to $u \in \mathcal{E}^1_\omega$ by approximation. Thus $u^g \in \mathcal{E}^1_\omega$ and the action $\mathcal{E}^1_\omega \times \mathrm{Aut}_0(X) \to \mathcal{E}^1_\omega$ is strongly continuous. As $u \to u^g = \tau_g + g^*u$ is monotone increasing, $P_\omega(u^g, v^g) = P_\omega(u,v)^g$. Hence the action is an isometry on $\mathcal{E}^1_\omega$.
Finally, assume $\omega > 0$. Since $\mathrm{Aut}_0(X)$ acts by isometries, we can assume $u = 0$. By closedness of $\{g \in \mathrm{Aut}_0(X) \,:\, d_1(\tau_g, 0) \leq C\}$, it is enough to show that any sequence g_j such that $d_1(\tau_{g_j}, 0) \leq C$ admits a convergent subsequence in $\mathrm{Aut}(X)$. By [Darvas-Lu '20] $\Delta_\omega \tau_{g_j} \leq C'$.
Hence $g_j : X \to X$ is uniformly Lipschitz wrt (the Riemannian metric induced by) ω.

Action by automorphisms

As the neutral component $\mathrm{Aut}_0(X)$ acts trivially on cohomology, for each $g \in \mathrm{Aut}_0(X)$ there exists a unique $\tau_g \in C^\infty(X)$ such that

$$g^*\omega = \omega + dd^c\tau_g, \quad \mathrm{E}_\omega(\tau_g) = 0.$$

For each $u \in \mathrm{PSH}(X, \omega)$, we set $u^g := \tau^g + g^*u$. $\rightsquigarrow \omega + dd^c u^g = g^*(\omega_u)$.

Proposition

The right-action of $\mathrm{Aut}_0(X)$ on $\mathrm{PSH}(X, \omega)$ restricts to an isometry on $\mathcal{E}^1_\omega$, which preserves the Monge-Ampère energy. When $\omega > 0$, this action is further proper, i.e. $\{g \in \mathrm{Aut}_0(X) \, : \, d_1(u^g, u) \leq C\}$ is compact for any $u \in \mathcal{E}^1_\omega$ and $C > 0$.

Proof. Let $u \in \mathcal{E}^1_\omega \cap L^\infty(X)$. Then

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Given a closed Lie subgroup $G \subset \mathrm{Aut}_0(X)$, we define $d_{1,G}(u,v) := \inf_{g \in G} d_1(u^g, v)$ on $\mathcal{E}^1_\omega \times \mathcal{E}^1_\omega$. $\rightsquigarrow$ quotient pseudometric on $\mathcal{E}^1_\omega / G$, which is a metric when the action is proper.

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Consider a functional $M : \mathcal{F} \to \mathbb{R} \cup \{+\infty\}$ defined on a subset $\mathcal{F} \subset \mathcal{E}^1_\omega$, such that both M and $\mathcal{F}$ are translation invariant, and assume that $\mathcal{F}$ is G-invariant.

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Definition

We say that M is coercive modulo G if there exists $\delta > 0$, $C > 0$ such that $M \geq \delta J_G - C$ on $\mathcal{F}$, where $J_G(u) := \inf_{g\in G} J(u^g) := \inf_{g\in G} \left(\int_X u^g \omega^n - \mathrm{E}_\omega(u^g)\right)$.

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Lemma

M is coercive if and only if there exists $\delta' > 0$, $C' > 0$ such that $M(u) \geq \delta' d_{1,G}(u, 0) - C'$ for any $u \in \mathcal{F}^0 := \{ v \in \mathcal{F} \,;\, \mathrm{E}_\omega(v) = 0 \}$.

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Proof. Set $T_\omega := \sup_{u \in \mathrm{PSH}(X,\omega)} \left\{ \sup_X u - V_\omega^{-1} \int_X u\, \omega^n \right\} \in [0, +\infty)$, and let $u \in \mathcal{E}^1_\omega$ such that $\mathrm{E}_\omega(u) = 0$.

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$$J(u^g) = \int_X u^g\,\omega^n \leq \int_X |u^g|\left(\mathrm{MA}_\omega(u^g) + \mathrm{MA}_\omega(0)\right) = I_1(u^g, 0) \approx d_1(u^g, 0),$$

from which we get $J_G(u) \lesssim d_{1,G}(u,0)$.

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Consider a functional $M : \mathcal{F} \to \mathbb{R} \cup \{+\infty\}$ defined on a subset $\mathcal{F} \subset \mathcal{E}^1_\omega$, such that both M and $\mathcal{F}$ are translation invariant, and assume that $\mathcal{F}$ is G-invariant.

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M is coercive if and only if there exists $\delta' > 0$, $C' > 0$ such that $M(u) \geq \delta' d_{1,G}(u,0) - C'$ for any $u \in \mathcal{F}^0 := \{v \in \mathcal{F}\,;\, \mathrm{E}_\omega(v) = 0\}$.

Proof. Set $T_\omega := \sup_{u\in \mathrm{PSH}(X,\omega)} \left\{\sup_X u - V_\omega^{-1} \int_X u\,\omega^n\right\} \in [0,+\infty)$, and let $u \in \mathcal{E}^1_\omega$ such that $\mathrm{E}_\omega(u) = 0$. Then

$$J(u^g) = \int_X u^g\,\omega^n \leq \int_X |u^g| \left(\mathrm{MA}_\omega(u^g) + \mathrm{MA}_\omega(0)\right) = I_1(u^g,0) \approx d_1(u^g,0),$$

from which we get $J_G(u) \lesssim d_{1,G}(u,0)$. On the other hand

$$d_1(u^g,0) \leq d_1(u^g, \sup_X u^g) + d_1(\sup_X u^g, 0) = V_\omega \sup_X u^g + V_\omega |\sup_X u^g| - \mathrm{E}_\omega(u^g) = 2V_\omega \sup_X u^g,$$

where we used that $\sup_X u^g \geq \mathrm{E}_\omega(u^g) = 0$. Hence $d_1(u^g,0) \leq 2V_\omega T_\omega + 2J(u^g)$.

Strong convergence and strong compactness for varying classes

Let $\{\omega_j\}_{j\in\mathbb{N}}$ be a sequence of semipositive, big $(1,1)$-forms on X converging smoothly to a big semipositive form ω. Assume also that $\omega_j \geq (1-\varepsilon_j)\omega$ with $\varepsilon_j \to 0$.

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Set $V_j := V_{\omega_j}, V := V_\omega, \mathrm{E}_j := \mathrm{E}_{\omega_j}, \mathrm{E} := \mathrm{E}_\omega, \mathcal{E}^1_j := \mathcal{E}^1_{\omega_j}, \mathcal{E}^1 := \mathcal{E}^1_\omega, \mathrm{MA}_j := \mathrm{MA}_{\omega_j}, \mathrm{MA} := \mathrm{MA}_\omega$, and $d_{1,j}, d_1$ for the Darvas metrics.

Let $\{\omega_j\}_{j\in\mathbb{N}}$ be a sequence of semipositive, big $(1,1)$-forms on X converging smoothly to a big semipositive form ω. Assume also that $\omega_j \geq (1-\varepsilon_j)\omega$ with $\varepsilon_j \to 0$.

Definition

We say that a sequence $\{u_j\}_j$ with $u_j \in \mathrm{PSH}(X,\omega_j)$ converges weakly to $u \in \mathrm{PSH}(X,\omega)$ if $u_j \to u$ in L^1.

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Rmks:

- Since $\omega_j \leq C\omega_X$, this is equivalent to the convergence in $\mathrm{PSH}(X, C\omega_X)$.
- In particular, it is not hard to check that there exists $A > 0$ such that

$$\sup_X u - A \leq V_j^{-1} \int_X u\, \omega_j^n \leq \sup_X u$$

for any $u \in \mathrm{PSH}(X,\omega_j)$. Hence any sequence $u_j \in \mathrm{PSH}(X,\omega_j)$ with $\int_X u_j\, \omega_j^n$ uniformly bounded admits a subsequence u_{j_k} that converges weakly to $u \in \mathrm{PSH}(X,\omega)$. [This follows from the weak compactness of $\{u \in \mathrm{PSH}(X, C\omega_X) \,:\, |\sup_X u| \leq C\}$]

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Proposition

For any weakly convergent sequence $\mathrm{PSH}(X,\omega_j) \ni u_j \to u \in \mathrm{PSH}(X,\omega)$ we have $\limsup_{j\to+\infty} \mathrm{E}_j(u_j) \leq \mathrm{E}(u)$.

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Proof. We can assume $u_j \leq 0$. Indeed we know that $\tilde{u}_j := u_j - \sup_X u_j \to \tilde{u} := u - \sup_X u$. Moreover

$$\mathrm{E}_j(u_j) - \mathrm{E}_j(\tilde{u}_j) = V_j \sup_X u_j \longrightarrow V \sup_X u = \mathrm{E}(u) - \mathrm{E}(\tilde{u}).$$

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Proof. We can assume $u_j \leq 0$. We can assume $u_j \geq -C$ uniformly. This follows from $u_j^k := \max(u_j, -k) \to u^k := \max(u,-k)$, $\mathrm{E}_j(u_j) \leq \mathrm{E}_j(\max(u_j,-k))$ and $\mathrm{E}(u^k) \searrow \mathrm{E}(u)$.

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Proof. We can assume $u_j \leq 0$. We can assume $u_j \geq -C$ uniformly. Then

$$\mathrm{E}_j(u_j) \leq \mathrm{E}_j\left((1-\varepsilon_j)u\right) + \int_X \left(u_j - (1-\varepsilon_j)u\right)\left(\omega_j + (1-\varepsilon_j)dd^c u\right)^n.$$

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We have $\mathrm{E}_j\left((1-\varepsilon_j)u\right) \to \mathrm{E}(u)$. This easily follows from the smooth convergence $\omega_j \to \omega$ and the fact that u is bounded, as

$$\mathrm{E}_j\left((1-\varepsilon_j)u\right) = \frac{1}{n+1}\sum_{k=0}^{n}\int_X (1-\varepsilon_j)u\left(\omega_j + (1-\varepsilon_j)dd^c u\right)^k \wedge \omega_j^{n-k}.$$

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We have $\mathrm{E}_j\left((1-\varepsilon_j)u\right) \to \mathrm{E}(u)$. Similarly $\int_X u\left(\omega_j + (1-\varepsilon_j)dd^c u\right)^n \to \int_X u\,(\omega + dd^c u)^n$.

Let $\{\omega_j\}_{j\in\mathbb{N}}$ be a sequence of semipositive, big $(1,1)$-forms on X converging smoothly to a big semipositive form ω. Assume also that $\omega_j \geq (1-\varepsilon_j)\omega$ with $\varepsilon_j \to 0$.

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Proof. We can assume $u_j \leq 0$. We can assume $u_j \geq -C$ uniformly. Then

$$\mathrm{E}_j(u_j) \leq \mathrm{E}_j\left((1-\varepsilon_j)u\right) + \int_X \left(u_j - (1-\varepsilon_j)u\right)\left(\omega_j + (1-\varepsilon_j)dd^c u\right)^n.$$

We have $\mathrm{E}_j\left((1-\varepsilon_j)u\right) \to \mathrm{E}(u)$. Similarly $\int_X u\left(\omega_j + (1-\varepsilon_j)dd^c u\right)^n \to \int_X u\,(\omega + dd^c u)^n$. On the other hand

$$\int_X u_j\left(\omega_j + (1-\varepsilon_j)dd^c u\right)^n = \int_X u_j\left(\omega_j - (1-\varepsilon_j)\omega + (1-\varepsilon_j)(\omega + dd^c u)\right)^n \leq (1-\varepsilon_j)^n \int_X u_j(\omega + dd^c u)^n$$

as $u_j \leq 0$ and $\omega_j \geq (1-\varepsilon_j)\omega$. The proof concludes setting $v_j := \left(\sup_{k\geq j} u_k\right)^*$ and observing that, by Monotone Convergence,

$$\int_X u_j\,(\omega + dd^c u)^n \leq \int_X v_j\,(\omega + dd^c u)^n \to \int_X u\,(\omega + dd^c u)^n$$

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- Let $u \in \mathcal{E}^1 \cap L^\infty(X)$. Then $\mathcal{E}^1_j \ni u_j := (1-\varepsilon_j)u \to u$ strongly.
- if $u \in \mathcal{E}^1 \cap L^\infty(X)$ and $u_j \in \mathcal{E}^1_j$ decreases to u, then $u_j \to u$ strongly.

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We say that a sequence $\{u_j\}_{j\in\mathbb{N}}$ with $u_j \in \mathcal{E}_j^1$ converges strongly to $u \in \mathcal{E}^1$ if it converges weakly and $\mathrm{E}_j(u_j) \to \mathrm{E}(u)$.

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- Let $u \in \mathcal{E}^1 \cap L^\infty(X)$. Then $\mathcal{E}_j^1 \ni u_j := (1-\varepsilon_j)u \to u$ strongly.
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If $u_j, v_j \in \mathcal{E}_j^1$ converge stronlgy to $u, v \in \mathcal{E}^1$, then $d_{1,j}(u_j, v_j) \to d_1(u, v)$.

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$$d_{1,j}(u_j, v_j) \leq d_{1,j}\left(u_j, (1-\varepsilon_j)u^k\right) + d_{1,j}\left((1-\varepsilon_j)u^k, (1-\varepsilon_j)v^k\right) + d_{1,j}\left((1-\varepsilon_j)v^k, v_j\right).$$

We claim that $\limsup_{j\to+\infty} \mathrm{RHS} \leq d_1(u, u^k) + d_1(u^k, v^k) + d_1(v^k, v) \xrightarrow{k\to+\infty} d_1(u, v)$.

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$$d_{1,j}(u_j, v_j) \leq d_{1,j}\left(u_j, (1-\varepsilon_j)u^k\right) + d_{1,j}\left((1-\varepsilon_j)u^k, (1-\varepsilon_j)v^k\right) + d_{1,j}\left((1-\varepsilon_j)v^k, v_j\right).$$

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$$d_{1,j}\left(u_j, (1-\varepsilon_j)u^k\right) \leq d_{1,j}\left(u_j, u_j^k\right) + d_{1,j}\left(u_j^k, (1-\varepsilon_j)u^k\right) = 2\mathrm{E}_j(u_j^k) - \mathrm{E}_j(u_j) - \mathrm{E}_j\left((1-\varepsilon_j)u^k\right).$$

We deduce $\limsup_{j\to+\infty} d_{1,j}\left(u_j, (1-\varepsilon_j)u^k\right) \leq d_1(u, u^k)$ as $u_j \to u$ strongly and $(1-\varepsilon_j)u^k \to u^k$ strongly.

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$$d_{1,j}(u_j, v_j) \leq d_{1,j}\left(u_j, (1-\varepsilon_j)u^k\right) + d_{1,j}\left((1-\varepsilon_j)u^k, (1-\varepsilon_j)v^k\right) + d_{1,j}\left((1-\varepsilon_j)v^k, v_j\right).$$

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$$\begin{aligned} d_{1,j}\left((1-\varepsilon_j)u^k, (1-\varepsilon_j)v^k\right) &\leq d_{1,j}\left((1-\varepsilon_j)u^k, (1-\varepsilon_j)w^k\right) + d_{1,j}\left((1-\varepsilon_j)w^k, (1-\varepsilon_j)v^k\right) \\ &= \mathrm{E}_j\left((1-\varepsilon_j)u^k\right) + \mathrm{E}_j\left((1-\varepsilon_j)v^k\right) - 2\mathrm{E}_j\left((1-\varepsilon_j)w^k\right) \xrightarrow{j\to+\infty} d_1(u^k, v^k). \end{aligned}$$

This proves the claim and concludes the first part of the proof.

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Since $\liminf_{j\to+\infty} d_{1,j}(u_j, w_j) = \liminf_{j\to+\infty} \big(\mathrm{E}_j(u_j) - \mathrm{E}_j(w_j)\big) \geq \mathrm{E}(u) - \mathrm{E}(w) = d_1(u, w)$, and similarly $\liminf_{j\to+\infty} d_{1,j}(w_j, v_j) \geq d_1(w, v)$, we get

$$d_1(u, v) \leq d_1(u, w) + d_1(w, v) \leq \liminf_{j\to+\infty} \big(d_{1,j}(u_j, w_j) + d_{1,j}(w_j, v_j)\big) = \liminf_{j\to+\infty} d_{1,j}(u_j, v_j).$$

Fix a volume form ν on X and denote by

$$H_j(u) := \frac{1}{2}\mathrm{Ent}\left(\mathrm{MA}_j(u)|\nu\right), \quad H(u) := \frac{1}{2}\mathrm{Ent}\left(\mathrm{MA}(u)|\nu\right)$$

respectively the *entropy functions* on $\mathcal{E}^1_j$ and on $\mathcal{E}^1$.

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Recall: Given two positive measures μ, ν,

$$\mathrm{Ent}(\mu|\nu) := \begin{cases} \int_X f \log f \, d\nu & \text{if } d\mu = f d\nu \\ +\infty & \text{otherwise.} \end{cases}$$

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Theorem

Any sequence $\{u_j\}_{j \in \mathbb{N}}$ such that $\sup_X u_j$ and the entropy $H_j(u_j)$ are both bounded admits a subsequence that converges strongly to some $u \in \mathcal{E}^1$.

Lemma

There exists a uniform constant $C > 0$ such that $\mathrm{E}_j(u) \geq V_j \sup_X u - C\big(H_j(u) + 1\big)$ for all $u \in \mathcal{E}^1_j$ and all $j \in \mathrm{N}$.

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Proof. By Zeriahi '01 there exists $\alpha > 0$, $B > 0$ such that

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$$-\alpha \mathrm{E}_j(u) \leq \int_X (-\alpha u)\mathrm{MA}_j(u) \leq 2H_j(u) + V_j \log \int e^{-\alpha u} d\nu - V_j \log V_j \leq 2H_j(u) + C$$

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Sketch of the Proof. Assume $f = 0$ (one can reduce to this case). As $\int_X e^{|f_j|} d\nu$ is bounded, the sequence $|f_j|^2$ is uniformly integrable. Thus, as $f_j \to 0$ ν-a.e., we deduce that $\int_X |f_j|^2 d\nu \to 0$.

Lemma

There exists a uniform constant $C > 0$ such that $\mathrm{E}_j(u) \geq V_j \sup_X u - C\left(H_j(u) + 1\right)$ for all $u \in \mathcal{E}^1_j$ and all $j \in \mathbb{N}$.

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$$|f_j| g \leq |f_j| e^{p|f_j|} + p^{-1} \chi(g).$$

This follows using the convex conjugate weights (on $\mathbb{R}_+$) $\chi(x) = (x+1)\log(x+1) - x$ and

$$\chi^*(y) := \sup_{x \geq 0} \{xy - \chi(x)\} = e^y - y - 1 \leq y e^y.$$

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Since $\chi(x) - x \log x$ is bounded from above on R_+, we have $\int_X \chi(g)\, d\nu \leq C'$ for C' that only depend on C.

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For any $\varepsilon > 0$ one can choose $p \gg 1$ such that $p^{-1}C' \leq \varepsilon$ to have that $\int_X |f_j| d\mu \leq 2\varepsilon$ when $\|f_j\|_{L^2(\nu)}$ is small enough.

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As elements in $\mathcal{E}^1_j, \mathcal{E}^1$ have zero Lelong numbers (Di Nezza-Darvas-Lu '18), the main result in Zeriahi '01 gives that

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Hence, from the Lemma below we get $\int_X (u_j - u)d\mu_j \overset{j\to+\infty}{\longrightarrow} 0$, $\sup_j \int_X |u - u^k| d\mu_j \overset{k\to+\infty}{\longrightarrow} 0$.

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As $\varepsilon_j \int_X u^k \, d\mu_j \geq -kV_j\varepsilon_j \overset{j\to+\infty}{\longrightarrow} 0$, we deduce that $\liminf_{j\to+\infty} \mathrm{E}_j(u_j) \geq \lim_{k\to+\infty} \mathrm{E}(u^k) = \mathrm{E}(u)$.

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Proof of the Theorem. As $\sup_X u_j$ is bounded, we may assume that $u_j \to u$ weakly. By the first Lemma, $\mathrm{E}_j(u)$ is bounded below. We deduce that $\mathrm{E}(u) \geq \limsup_{j\to+\infty} \mathrm{E}_j(u_j) > -\infty$, i.e. $u \in \mathcal{E}^1$. The second Lemma then concludes the proof.

Openness of coercivity

The general recipe for openness of coercivity

Fix a closed subgroup $G \subset \mathrm{Aut}_0(X)$ and consider functionals

$$M : \mathcal{F} \to \mathbb{R} \cup \{+\infty\}, \quad M_j : \mathcal{F}_j \to \mathbb{R} \cup \{+\infty\},$$

respectively defined on subsets of $\mathcal{E}^1$ and $\mathcal{E}^1_j$ and satisfying the following conditions:

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respectively defined on subsets of $\mathcal{E}^1$ and $\mathcal{E}^1_j$ and satisfying the following conditions:

- Invariance: both $\mathcal{F}_j, \mathcal{F}$ and M_j, M are invariant under translation and under the action of G;
- Normalization: 0 lies in $\mathcal{F}_j$ and $\mathcal{F}$, and $M_j(0) \to M(0)$.
- Lower Semicontinuity: If $\mathcal{F}_j \ni u_j \to u \in \mathcal{E}^1$ strongly, then $u \in \mathcal{F}$ and $\liminf_{j \to +\infty} M_j(u_j) \geq M(u)$.
- Convexity: $\mathcal{F}_j$ is convex wrt psh geodesics and M_j is convex along such geodesics.
- Entropy Growth: there exists $\delta > 0$, $C > 0$ such that $M_j(u) \geq \delta H_j(u) - C\left(d_{1,j}(u, 0) + 1\right)$ for any $u \in \mathcal{F}_j$ and any $j \in \mathbb{N}$.
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Theorem

Set $\mathcal{F}^0 := \{u \in \mathcal{F} \,:\, \mathrm{E}(u) = 0\}$ and $\mathcal{F}^0_j := \{u \in \mathcal{F}_j \,:\, \mathrm{E}_j(u) = 0\}$. Suppose that there exists $\delta, C \in \mathbb{R}$ (δ is not necessarily positive!) such that

$$M(u) \geq \delta d_{1,G}(u,0) - C$$

for any $u \in \mathcal{F}^0$. Then for any $\delta' < \delta$ there exists $C' \in \mathbb{R}$ and $j_0 \in \mathbb{Z}_{\geq 0}$ such that

$$M_j(u) \geq \delta' d_{1,j,G}(u,0) - C'$$

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Let $(u_{j,t})_{t\in[0,T_j]} \subset \mathcal{F}_j^0$ be the unit-speed geodesic joining 0 and u_j.

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$$t\delta' \geq M(v_t) \geq \delta d_{1,G}(v_t, 0) - C = \delta d_1(v_t, 0) - C = \delta t - C,$$

which leads to a contradiction when $t > C/(\delta - \delta')$.

Thank you for your attention!