BJT : section 3-8

<u>Goal</u> : 1/ defined singular / degenerate object on X smooth
2/ New estimates for weighted energy functionals

<u>1.</u> Equi form / current and moment polytope

(X, ω_X, T) $\quad T \subset \mathrm{Aut}_{red}(X)$

<u>$\forall\, \omega$ closed</u> $(1,1)$-form T-inv:

$\exists\, m_\omega : X \longrightarrow \mathfrak{t}^*$

$\mathfrak{t} = \mathrm{Lie}(T)$

$$\omega(\xi, \cdot) = -\, d\, m_\omega^{\xi}$$

$$m_\omega^{\xi} = \langle m_\omega, \xi \rangle_{\mathfrak{t}, \mathfrak{t}^*}$$

$$\int_X m_\omega^{\xi}\, \omega^n = 0 \quad \text{"centered"} \quad \forall \xi$$

$\Omega := (\omega, m) \to$ equiv. form

$m_\Omega := m$

- θ closed $(1,1)$-current T-inv.

$\theta = \underset{\downarrow}{\omega} + dd^c f$, f distribution T-inv.

T-inv. form

distribution

$m_f^\xi := -df(J\xi)$ is a moment map for $dd^c f$

$\langle df, u \rangle = \langle f, du \rangle$

$\langle J\xi \lrcorner\, df, u \rangle = \langle df, J\xi \lrcorner\, u \rangle$

1 $\qquad$ $n-1$

$\langle df(J\xi), u \rangle$

$\shortparallel$

$-\langle f, d^c u(\cdot\xi) \rangle$

$\Theta := (\theta, m_\theta)$ equiv. current

$m_\Theta = m_\omega + m_f$

* ν volume form on X, T-inv

$$\mathrm{Ric}^T(\nu) = (\mathrm{Ric}(\nu), m_\nu)$$

$$\mathrm{Ric}(\nu) = -\frac{1}{2} dd^c \log(\nu)$$

$$m_\nu^\xi := \frac{\mathcal{L}_{J\xi}\nu}{2\nu}$$

$\rightarrow$ Ric^T equiv. Ricci form

<u>Lemma</u>: ω closed $(1,1)$-form T-inv, $\{\omega\}^n \neq 0$

1) (ω_j) of closed T-inv $(1,1)$-form $\xrightarrow[\mathcal{C}^\infty]{} \omega$, then

$$m_{\omega_j} \xrightarrow{\mathcal{C}^\infty} m_\omega$$

2) If ω is moreover semi-positive, then

$m_\omega(X)$ only depends on $\{\omega\}$

Proof:

1) $-\mathcal{L}_{J\xi}\,\omega_j$

$$= -d(\omega_j(J\xi,\cdot))$$

$$= -d(\omega_j(\xi, J\cdot))$$

$$= -d^c(\omega_j(\xi,\cdot))$$

$$= -d^c d\, m^\xi_{\omega_j} = dd^c m^\xi_{\omega_j}$$

$$\Rightarrow \quad -\Delta_{\omega_X}(\mathcal{L}_{J\xi}\omega_j) = -\underline{\Delta_{\omega_X} m^\xi_{\omega_j}}$$

$$\Delta_{\omega_X}(\lim m^\xi_{\omega_j} + m^\xi_\omega) \underset{?}{=} 0 \qquad \downarrow$$

$$\Rightarrow\ c_j + m^\xi_{\omega_j} \to m^\xi_\omega - \quad -\Delta_{\omega_X}(\mathcal{L}_{J\xi}\omega) = -\underline{\Delta_{\omega_X} m^\xi_\omega}$$

$$\Rightarrow \quad (m^\xi_{\omega_j} + c_j) \xrightarrow{\mathcal{C}^\infty} m^\xi_\omega$$

$$c_j\{\omega_j\}^n = \int_X (m^\xi_{\omega_j} + c_j)\,\omega_j^n$$

$$\downarrow$$

$$c\,\{\omega\}^n = \int_X m^\xi_\omega\, \omega^n = 0$$

$$\Rightarrow \quad c = 0$$

2/ $\omega > 0$ Kähler.

$m_\omega(X) = \operatorname{supp} DH_{m_\omega}$

$(m_\omega)_*(\omega^n)$

$v \in C^\infty(P)$

$$\int_X v(m_\omega)\,\omega^n = \int_P v\, DH_{m_\omega}$$

$$\underbrace{m_{\omega + \varepsilon\omega_X}}_{>0} \xrightarrow[\varepsilon \to 0]{C^\infty} m_\omega$$

$\square$

2 - Weighted Monge-Ampere operator and weighted energy:

def°:

- v - weighted MA operator

$$MA_{\Omega,v}(\varphi) := v(m_{\Omega_\varphi})\,\omega_\varphi^n$$

$$\Omega_\varphi := \Omega + dd^c_T \varphi$$

$$MA_{\Omega,v} : K \to \Omega^{n,n}\text{-form}$$

• Θ equiv. current

Θ-twisted v-weighted MA operator $\mathrm{MA}(\Omega+t\Theta), v$

$$\mathrm{MA}^{\Theta}_{\Omega,v}(\varphi) = v(m_{\Omega\varphi})\,\Theta\wedge\omega_\varphi^{n-1} + \langle v'(m_{\Omega\varphi}), m_\Theta\rangle\,\omega_\varphi^{n}$$

(*)

It follows by IBP argument that MA_v and MA^{Θ}_v admits Euler–Lagrange fctional

def$^{\underline{n}}$:

1) $E_{\Omega,v} : \mathcal{C}^\infty(X)^T \longrightarrow \mathbb{R}$

EL of MA_v

2) $E^{\Theta}_{\Omega,v} : \mathcal{C}^\infty(X)^T \longrightarrow \mathbb{R}$

EL of MA^{Θ}_v

More concretely:

$E_v(\varphi) - E_v(\psi)$ (*)

$$= \int_0^1 dt \int_X \dot\varphi_t\, v(m_{\Omega\varphi_t})\,\omega_{\varphi_t}^n$$

ℓ_t path joining φ and ψ. $\Omega = (\omega, \omega)$

$\mathcal{H}^T = \{ T\text{-inv psh in } [\omega] \text{ rel to } \omega \}$

$\underset{\cup}{\mathcal{E}^{1,T}} = \overline{\mathcal{H}^T}^{,d_1}$ d_1 Darvas distance

$\{ PSH(\omega) \text{ with finite energy} \}$

<u>prop$^{\circ}$</u>: θ smooth

1) E_v / E_v^θ admits $\mathcal{C}^0$ extension to $\mathcal{E}^{1,T}$ s.t. $A>0$ cte

1) $|E_v(\varphi) - E_v(\psi)| \leq A\, d_1(\varphi,\psi)$

2) $|E_v^\theta(\varphi) - E_v^\theta(\psi)|$
$$\leq B \, \underbrace{d_1(\varphi,\psi)^\alpha \max\{d_1(\varphi,0), d_1(\psi,0)\}^{1-\alpha}}$$
$\alpha = 2^{-n}$

$$MA^\theta_{\Omega,v}(\varphi) = \frac{d}{dt}\Big|_{t=0} MA_{\Omega+t\theta,v}(\varphi)$$

Pre

Proof:

1) $(*) \Rightarrow |E_v(\varphi) - E_v(\psi)|$

$$\leq \sup_p |v| \; d_1(\varphi, \psi)$$

2) $|E_v^\theta(\varphi) - E_v^\theta(\psi)|$

φ_t line joining φ and ψ

$$= \int_0^1 dt \int_X (\varphi - \psi) \Big[v(m_{\Omega \varphi_t}) \, n \, \theta \wedge \omega_{\varphi_t}^{n-1} + \langle v'(m_{\Omega \varphi_t}), m_\theta \rangle \Big] \omega_{\varphi_t}^{n}$$

$\varphi_t = t\varphi + (1-t)\psi$

$$= \sum_{p=0}^{n-1} \int_X a_p (\varphi - \psi) \, \theta \wedge \omega_\varphi^{p} \wedge \omega_\psi^{n-1-p} \qquad (I)$$

$$+ \sum_{p=0}^{n} \int_X b_p (\varphi - \psi) \, \omega_\varphi^{p} \wedge \omega_\psi^{n-1-p}$$

a_p and b_p is bdd

$$(I) \leq c \int_X |\varphi - \psi| \, \omega \wedge \omega_\varphi^{p} \wedge \omega_\psi^{n-1-p}$$

$$\begin{pmatrix}\text{Thm}\\ 1.2\end{pmatrix} \to \leq C \, d_1(\varphi, \psi)^{\alpha} \max\{ d_1(\varphi,0), d_1(\psi,0)\}^{1-\alpha}$$

$$C = (\|\theta\|_{\omega}, \sup(v), n)$$

(II) is similar.

3- Weighted Scalar curvature and weighted Mabuchi energy:

def$^{\underline{c}}$: θ equiv. current

- $\mathrm{tr}_{\Omega, v}(\theta) = MA_v^{\theta}(\theta)$
- $\mathrm{Ric}_{C_v}^{T}(\Omega) = \underline{\mathrm{Ric}_C^{T}(MA_v(\theta))}$

v - Ricci form

- $S_v(\Omega) = \underline{\mathrm{tr}_{\Omega, v}(\mathrm{Ric}_v^{T}(\Omega))}$

v - Scalar curvature

$$S_v^{\mathrm{Lah}} = v \, S_v$$

• $v, w > 0$, $\exists!\ \ell^{ext}$ affine fc$^{\underline{c}}$ on $\mathfrak{t}^*$ /

$$\langle \ell, \ell^{ext} \rangle = \int_X \ell(m_\Omega)\, MA_v(0)$$

$\forall\ \ell$ affine.

$$\langle \ell', \ell \rangle := \int_X \ell(m_\Omega)\, \ell'(m_\Omega)\, MA_{vw}(0)$$

$\llcorner$, weighted Futaki--Mabuchi pairing

<u>def$^{\underline{c}}$</u>: $v, w > 0$, Ω is said (v,w)-extremal if

$\llcorner$ $S_v(\Omega) \underset{?}{=} w(m_\Omega)\, \ell^{ext}(m_\Omega)$

Moreover (v,w)-extremal metric are critical point for $\sim \mathrm{Ric}^T(\gamma)$

$$M^{rel}_{v,w}(\varphi) := H_v(\varphi) + \underline{E_v}\ (\varphi)$$

$+ \ \underline{Evw_{l,ext}}(\varphi)$

- $H_\nu(\varphi) = \frac{1}{2} \operatorname{Ent}(MA_\omega(\varphi) \,|\, \nu)$

$= \frac{1}{2} \int_X \log\left(\frac{MA_\omega(\varphi)}{\nu}\right) MA_\omega(\varphi)$

$M^{rel}_{\nu,w}$ is l.s.c. on $\mathcal{E}^{1,T}$

$T \subset_{max} \operatorname{Aut}_{red}(X)$, $\underline{\nu \text{ log concave}}$

- **Thm**: $\exists$ $\omega_\varphi \in [\omega]$ $(\underline{\nu},\underline{w})$-extremal

metric iff $\exists\, C, D > 0$ $\Downarrow \forall \nu$

$M^{rel}_{\nu,w}(\varphi) \geq \underline{C}\ \underline{d_{1,T^{\mathbb{C}}}}(0,\varphi) - D$

(coercivity) $\varphi \in \mathcal{K}^T$

$d_{1,T^{\mathbb{C}}}(0,\varphi) = \inf_{\gamma \in T^{\mathbb{C}}} d_1(0, \gamma\cdot\varphi)$. $\square$

$E_\nu^{\theta^{\mathbb{C}}}$ $\quad \theta = \omega + dd^c f$

$\boxed{E_\nu^{dd^c f}}$: $\mathcal{E}^{1,T} \longrightarrow \mathbb{R} \cup \{-\infty\}$

that is $\mathcal{C}^0$ along $\downarrow$ sequences

(X fano , v-soliton

$$M_v = H_v - (I_v - J_v) + e^f$$

$$\geq \delta_v^{\mathbb{T}^c}(I_v - J_v) \sim (I_v - J_v) - C$$

$$\geq (\delta_v^{\mathbb{T}^c} - 1) \inf_{g \in T^c} (I_v - J_v)(g \cdot \varphi) - C$$

$\cdot$ $[\omega] = c_1(X) + [\theta]$

X Kähler   Kähler