X: n-$\dim_{\mathbb{C}}$ complex analytic space. ie
$\forall x \in X$, $\exists U \subset X$, $U \ni x$, $j: U \to \mathbb{C}^N$
$j(U) = \{f_1 = \dots = f_n = 0\}$. f_i holo.

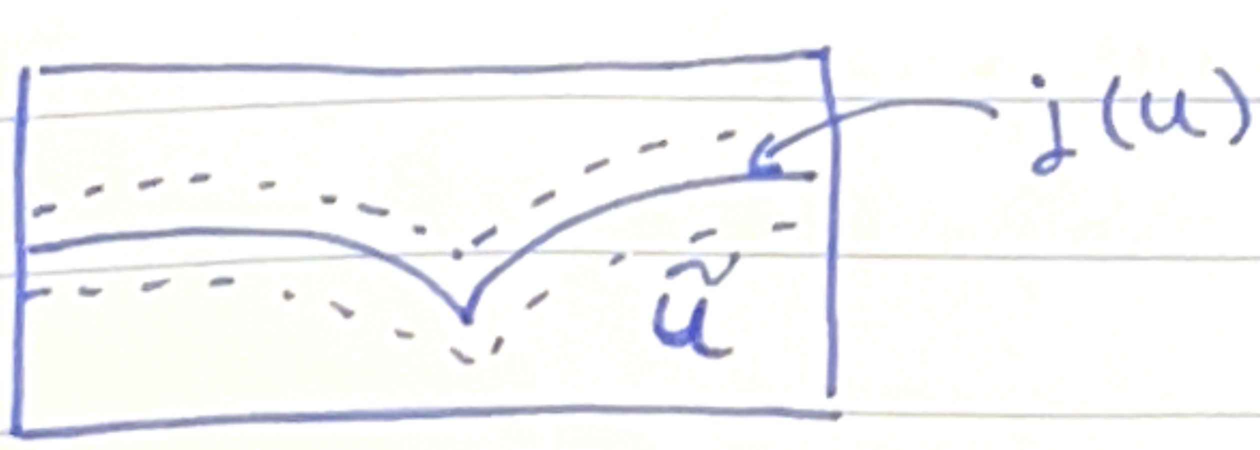
Assume that X is Kähler & normal

<u>Normal:</u> $\mathcal{O}_X \simeq \tilde{\mathcal{O}}_X$ ← sheaf of bounded & holo fcts on X_{reg}.

(<u>Def:</u> $\tilde{\mathcal{O}}_X$
A weakly holo fcts on X is a holo fct on X_{reg}, s.t
$\forall x \in X_{sing}$, $\exists V \ni x$: f is bounded on $X_{reg} \cap V$.
$\tilde{\mathcal{O}}_{X,x}$ the ring of germs of weakly holo fcts over nbd of x, $\tilde{\mathcal{O}}_X$ the associated sheaf $\}$.
<u>Def:</u> Normal.
X normal: $\forall x \in X$, $\mathcal{O}_{X,x} \simeq \tilde{\mathcal{O}}_{X,x}$).

By def of complex spaces ($\forall \mathcal{U}$: $\mathcal{O}_{\mathbb{C}(\tilde{\mathcal{U}})} \simeq \mathcal{O}_{X|_{\mathcal{U}}}$)

$\forall x \in X$, $(f)_x \in \mathcal{O}_{X,x}$: $\exists \underset{x}{\underset{\cap}{U}} \to \mathbb{C}^N$ $f = F|_{j(U)}$
where $F \in \mathcal{O}_{\mathbb{C}^N}(\tilde{\mathcal{U}})$
$\tilde{\mathcal{U}}$ nbd of U in $\mathbb{C}^N$.



<u>Normal.</u> $\Rightarrow$ $\mathcal{O}_X \simeq \iota_* \mathcal{O}_{X_{reg}}$, $\iota: X_{reg} \to X$.
- $\Rightarrow$ $\operatorname{codim} X_{sing} \geq 2$
- $\Rightarrow$ $H^0(X, TX) \simeq H^0(X_{reg}, TX_{reg})$.

Let $\xi \in H^0(X, TX)$.

Fact we can view ξ as a holo v.f on a nbd of $j(\mathcal{U})$. $\quad j : \mathcal{U} \to \mathbb{C}^N$ local embedding.

$x \in X_{reg}$, $(x^1, \dots, x^n)$ local coord around x.

$$\xi = f^i \partial_i = \underbrace{f^i \frac{\partial z^j}{\partial x^i}}_{=: F_j} \frac{\partial}{\partial z^j}$$

$F_j \in \mathcal{O}_{\mathcal{U}_{reg}} \simeq \mathcal{O}_{\mathcal{U}}$ since X is normal.

$\Rightarrow$ ξ extend to a nbd of $j(\mathcal{U})$.

Lemma:

$\forall \xi \in H^0(X, TX)$, ω smooth form of X

$\iota_\xi \omega$, $\mathcal{L}_\xi \omega$

are smooth forms.

Pf:

$j : \mathcal{U} \to \mathbb{C}^N$.

$\omega = \tilde{\omega}|_{j(\mathcal{U})}$, $\quad \xi = \tilde{\xi}|_{j(\mathcal{U})}$

$\Rightarrow \iota_\xi \omega = (\iota_{\tilde{\xi}} \tilde{\omega})|_{j(\mathcal{U})}$

Moreover $\mathcal{L}_\xi \omega = d\iota_\xi \omega + \iota_\xi d\omega$

for θ smooth form: $\theta = \tilde{\theta}|_{j(\mathcal{U})} \Rightarrow d\theta = d\tilde{\theta}|_{j(\mathcal{U})}$. $\square$

Denote Z (resp Z') the space of locally dd^c exact real $(1,1)$ forms (resp currents) on X. Any $\vartheta \in Z'$:

$$\vartheta = \omega + dd^c f$$

w/ $\omega \in Z$, f a distribution.

($T = dd^c \varphi_\alpha$, U_α cover of X, (χ_α) partition of unity.

On any U_α: $T = dd^c \varphi_\alpha$.
On $U_{\alpha\beta}$ $dd^c(\varphi_\alpha - \varphi_\beta) = 0 \Rightarrow \varphi_\alpha - \varphi_\beta$ is smooth.
Def: $f = \sum \chi_\alpha \varphi_\alpha$.

$$\vartheta = T - dd^c f \overset{\text{loc on } U_\beta}{=} dd^c \varphi_\beta - dd^c \sum_\alpha \chi_\alpha \varphi_\alpha$$

$$= dd^c \sum_\alpha \chi_\alpha \underbrace{(\varphi_\beta - \varphi_\alpha)}_{\text{smooth}} \sim \text{smooth}.$$

$\Rightarrow T = \vartheta + dd^c f.$)

Lemma:

$\forall g \in \mathrm{Aut}_0(X)$, $\omega \in Z$, $\exists \tau \in \mathcal{C}^\infty(X)$ s.t
$$g^* \omega = \omega + dd^c \tau.$$

Pf:

Let $\pi: \tilde{X} \to X$ a $\mathrm{Aut}_0(X)$ resol of sing, $\tilde{X}$ Kähler. (ie $\forall g \in \mathrm{Aut}_0(X)$, $\exists \tilde{g} \in \mathrm{Aut}_0(\tilde{X}) = \mathrm{Aut}_0(X)$ s.t $g \circ \pi = \pi \circ \tilde{g}$)

$$\Rightarrow [\tilde{g}^* \pi^* \omega] \underset{\text{equiv.}}{=} [\pi^* g^* \omega]$$

As Aut_0 acts trivially on H^2 $[g^* \pi^* \omega] = [\pi^* \omega]$
$\Rightarrow \pi^* g^* \omega = \pi^* \omega + dd^c \tilde{\tau}$, $\tilde{\tau} \in \mathcal{C}^\infty(\tilde{X})$

$\Rightarrow$ $\tilde{\tau}$ is constant along the fibers of π.
$\Rightarrow$ $\exists\, \tau \in C^0$ s.t $\tilde{\tau} = \pi^* \tau$.
$\Rightarrow$ $g^* \omega = \omega + dd^c \tau$.
as $g^*\omega$ & ω are smooth, $dd^c\tau$ and hence τ are smooth too.

Define the reduced aut grp:

$$\mathrm{Aut}_r(X) \subset \mathrm{Aut}_0(X)$$

as the identity comp of the subgrp of $\mathrm{Aut}_0(X)$ that acts trivially on the Albanese torus of some $\mathrm{Aut}_0(X)$-equiv resol of sing of X.
(by ~~inv~~ mero inv of Albanese torus it is indep of the resol).
($\pi: \tilde{X} \to X$ resol, Ab torus: $H^0(\tilde{X}, \Omega^1_{\tilde{X}})/H^1(\tilde{X}, \mathbb{Z})$)

Moment map:

Fix $T \subset \mathrm{Aut}_r(X)$ a compact torus /w Lie algebra $\mathfrak{t}$.

~~A moment map for a $\mathbb{Z}$-~~ ~~T-inv~~

Definition:

A moment map for a T-inv $\omega \in \mathcal{Z}$ (resp $\omega \in \mathcal{Z}'$) is a T-inv $\mathfrak{t}^*$ valued smooth map (resp. distribution) m on X s.t

$$\forall \xi \in \mathfrak{t}: \quad -dm^{\xi} = \iota_{\xi}\, \omega \qquad (m^{\xi} := \langle m, \xi \rangle).$$

We call $\Omega = (\omega, m)$ an equivariant form, with moment map $m_{\Omega} := m$.

Lemma:

Any T-inv form $\omega \in \mathcal{Z}$ (resp $\in \mathcal{Z}'$) admits a moment map, unique up to addition of a constant in t^*.

Pf:

Assume $\omega \in \mathcal{Z}$.

$\pi : \tilde{X} \to X$ T-~~inv~~ equiv resol of sing.

$\Rightarrow$ $\pi^* \omega$ admits a moment map on $\tilde{X}$.

$\Rightarrow$ $\exists$ smooth moment map m for ω on X_{reg}.

Need to show that m extends to a smooth map on X.

$\forall x \in X$, $x \in \mathcal{U}$: $\omega = dd^c \varphi$ on $\mathcal{U}$, $\varphi \in \mathcal{C}^\infty(\mathcal{U})$.

$\forall \xi \in t$:

on $\mathcal{U}_{reg}$: $-dd^c m^\xi = \mathcal{L}_{J\xi}\, \omega = dd^c \mathcal{L}_{J\xi} \varphi$.

Note $h = m^\xi + \mathcal{L}_{J\xi} \varphi$. $dd^c h = 0$ on $\mathcal{U}_{reg}$

~~$\Rightarrow$ h extends~~ ~~h is smooth on $\mathcal{U}_{reg}$~~

$\Rightarrow$ h extends smoothly on $\mathcal{U}$, as $\mathcal{L}_{J\xi}$ is smooth on $\mathcal{U}$ m^ξ also have a smooth extension on $\mathcal{U}$.

If $\omega \in \mathcal{Z}'$, $\omega = \omega' + dd^c f$, $\omega' \in \mathcal{Z}^T$, f a T-inv distrib.

ω' admits a smooth moment map m'.

$\Rightarrow$ $m = m' + m_f$ is a m.m for ω.

$(m_f^\xi = d^c f(\xi))$

Weighted MA, weighted energy:

$\Omega = (\omega, m_\Omega)$ equiv form,

$v \in \mathcal{C}^\infty(t^*)$.

Set: $\Theta = (\theta, m_\Theta)$ an equicurrent.

$$\Omega_\varphi := \Omega + dd^c_T \varphi = (\omega_\varphi, m_\Omega + m_\varphi)$$

for $\varphi \in \mathcal{C}^\infty(X)^T$,

$(dd^c_T f := (dd^c f, m_f))$.

Now we define as is the smooth setting:

$$MA_{\Omega,v}(\varphi) = v(m_{\Omega_\varphi})\, \omega_\varphi^n.$$

$$MA^\Theta_{\Omega,v}(\varphi) := v(m_{\Omega_\varphi})\, n\, \theta \wedge \omega_\varphi^{n-1} + \langle v'(m_{\Omega_\varphi}), m_\Theta \rangle \omega_\varphi^n$$

As in the smooth setting, $\forall \varphi \in \mathcal{C}^\infty(X)^T$, f, g T-inv distributions with at least one smooth.

$$\int_X g\, MA^{dd^c_T f}_{\Omega,v}(\varphi) = \int_X f\, MA^{dd^c_T g}_{\Omega,v}(\varphi)$$

So we define the weighted energy fctls

$$E_{\Omega,v}: \mathcal{C}^\infty(X)^T \to \mathbb{R}, \quad E^\Theta_{\Omega,v}: \mathcal{C}^\infty(X)^T \to \mathbb{R}$$

as the E-L fctls for the weighted MA op.

(ie $\frac{d}{dt}\Big|_{t=0} E(\varphi + t\psi) = \int \psi\, MA(\varphi)$)

Extension to $\mathcal{E}^1$:

Fix $\Omega = (\omega, m)$. $\mathcal{H} = \mathcal{H}(X, \omega) = \{\varphi \in \mathcal{C}^\infty \mid \omega_\varphi > 0\}$.

Thm (Cho–Choi + Richberg)

$\forall \varphi \in PSH(X, \omega)^{(T)}$: $\exists\, \varphi_j \in \mathcal{H}(X, \omega)^{(T)}$ s.t. $\varphi_j \searrow \varphi$

Estimates in the smooth setting apply without changes

$\Rightarrow$ extension of the MA op & energy to $\mathcal{E}^{1,T}$.

$\mathcal{E}^1$: potential w/ finite energy.

Adapted measures, volume forms.

Assume that X has l.t sing.

K_X is a $\mathbb{Q}$-line bundle.

Choose m s.t. mK_X is a line bundle, σ trivialization of mK_X.

Chose ψ a smooth metric on K_X.

Def

$$\mathcal{V} := \frac{\left(i^{mn^2}\, \sigma \wedge \bar{\sigma}\right)^{2/m}}{|\sigma|_\psi^{2/m}} \qquad (\text{l.t } \mathcal{V}(X_{reg}) < +\infty).$$

We call $\mathcal{V}$ an adapted measure, and denote $\frac{1}{2}\log \mathcal{V} := \psi$.

Define the Ricci form of $\mathcal{V}$:

$$\mathrm{Ric}(\mathcal{V}) = -\frac{\pi}{4}\, dd^c\, \frac{1}{2}\log \mathcal{V}$$

and the T equiv Ricci form:

$$\mathrm{Ric}^T(\mathcal{V}) := -dd^c_T\, \frac{1}{2}\log \mathcal{V}.$$

Lemma:

μ volume form on X, ν adapted measure.
Then $\mu = e^{\rho}\nu$, ρ quasi psh fct with analytic singularities. ie

$$\rho = c \max_{\alpha} |f_\alpha| + O(1)$$

for a finite set $f_\alpha \in \mathcal{O}_X$, $c \in \mathbb{Q}_{>0}$.

$\Rightarrow$ for μ a T-inv volume form we def

$$\mathrm{Ric}(\mu) = \underset{\substack{\uparrow \\ \text{smooth}}}{\mathrm{Ric}(\nu)} - \underset{\substack{\uparrow \\ \text{singular}}}{dd^c\rho}.$$

and the equiv Ricci current:

$$\mathrm{Ric}^T_{\xi}(\mu) = \mathrm{Ric}^T(\nu) - dd^c_T\rho.$$

Weighted scalar curvature

$$P := m_{\Omega}(X) \subset \mathfrak{t}^*$$

is a convex polytope.

Let $v \in C^\infty(\mathfrak{t}^*)$, positive on P. $\forall \varphi \in \mathcal{H}^T$ def the v weighted ~~Ri~~ equivariant Ricci current of Ω_φ

$$\mathrm{Ric}^T_v(\Omega_\varphi) = \mathrm{Ric}^T(MA_v(\varphi)).$$

and the scalar curvature

$$S_v(\Omega_\varphi) = \frac{MA^{\mathrm{Ric}^T_v(\Omega_\varphi)}(\varphi)}{MA_v(\varphi)}.$$

Pf of lemma: volume forms differ by a conf factor
2

$\Rightarrow$ assume $\mu = \omega^n$, $\omega = \sum_{k=1}^{N} i\, dz^i \wedge d\bar{z}^i$ for some local embedding $X \hookrightarrow \mathbb{C}^N$

σ local triv of mK_X. $\nu = |\sigma|^{2/m}$

$$\mu = \sum_{\substack{|I|=n \\ I \subset [1,n]}} i^{n^2} \sigma_I \wedge \bar{\sigma}_I, \quad \sigma_I = \bigwedge_{i \in I} dz^i$$

$\Rightarrow \forall I$, $\sigma_I^m = f_I \sigma$, f_I holo

$$\Rightarrow \mu = \sum_I |f_I|^{2/m} |\sigma|^{2/m}$$

$$\mu = \sum_I |f_I|^{2/m} \nu.$$

Given ν_X a T-inv adapted measure def weighted entropy & weighted Mabuchi Ricci energy
$$H_v : \mathcal{E}^{1,T} \to \mathbb{R} \cup \{+\infty\}, \quad R_v : \mathcal{E}^{1,T} \to \mathbb{R}$$
$$H_v(\varphi) = \frac{1}{2} \mathrm{Ent}(MA_v(\varphi) \mid \nu_X), \quad R_v(\varphi) = E_v^{-\mathrm{Ric}_v^T(\nu_X)} ?$$
$$\left(\mathrm{Ent}(MA_v(\varphi), \nu) = \frac{1}{2} \int \log\left(\frac{MA_v(\varphi)}{\nu_X}\right) MA_v(\varphi)\right).$$
Given a other $w \in \mathcal{C}^\infty(t^*)$
$$M_{v,w} : \mathcal{E}^{1,T} \to \mathbb{R} \cup \{+\infty\}$$
$$M_{v,w}(\varphi) = H_v(\varphi) + R_v(\varphi) + E_{vw}(\varphi).$$
The same computations as in the smooth case show that $M_{v,w}$ is a E.L fctl for
$$\varphi \mapsto (w(m_{\Omega_\varphi}) - S_v(\Omega_\varphi)) MA_v(\varphi).$$
$\mathcal{H}^{T} ?$

Assuming $w > 0$, def as in the smooth setting the extremal affine fct $\hat{\ell}^{ext}$ as the unique affine fct on t s.t
$$\int_X \ell(m_\Omega)\, \ell^{ext}(m_\Omega)\, w(m_\Omega)\, MA_v(0) = \int_X \ell(m_\Omega)\, S_v(\Omega)\, MA_v(0)$$
$\forall$ affine fct $\ell \in t \oplus \mathbb{R}$.