1. INTUITION

Theorem 1.1. *Suppose all the items in the principle. Suppose given $\sigma, C \in \mathbb{R}$ such that*

$$M(\varphi) \geq \sigma \ \mathrm{d}_1(\varphi, 0) - C$$

for all $\varphi \in \mathcal{E}^1_0 := \{\varphi \in \mathcal{E}^1 | \sup \varphi = 0\}$. Then, for any $\sigma' < \sigma$, there exist $C' \in \mathbb{R}$ and $j_0 \in \mathbb{Z}_{\geq 0}$ such that

$$M_j(\varphi) \geq \sigma' \ d_j(\varphi, 0) - C'$$

for any $\varphi \in \mathcal{E}^1_{j,0} := \{\varphi \in \mathcal{E}^1_j | \sup \varphi = 0\}$ and any $j \geq j_0$. Moreover, we assume the following items

1. *convexity of M_j*
2. *lower semi-continuity*
3. *entropy growth gives two estimates*
   - *$M_j + C(d(0, \varphi) + 1) \geq \delta H_j$ for some $\delta > 0$*
   - *Since H_j is uniformly bounded below, then $M_j \geq -C(d(0, \varphi) + 1)$ for C large enough.*
4. *normalization*

Proof. We proceed by contradition, there exist σ' such that for any C and any j_0 there is a $j \geq j_0$ such that $M_j(\varphi_j) \leq \sigma' d(0, \varphi_j) - C$. In particular, we assume

1. $M_j(\varphi_j) \leq \sigma' T_j - C_j$, where $C_j \to +\infty$ and $T_j := d(0, \varphi_j)$
2. Let $T_j \geq t > 0$, and $\varphi_{j,t}$ is the unit speed geodesic connecting 0 and $\varphi_{j,\cdot}$. By convexity of M_j along geodesics,
$$M_j(\varphi_{j,t}) \leq \frac{t}{T_j} M_j(\varphi_j) < t\sigma', \tag{1.1}$$
3. the entropy growth condition implies that $\sup_j \mathrm{H}_j(\varphi_{j,t}) < (\sigma' t + C'(t+1))/\delta$.and $\mathrm{d}_{1,j}(\varphi_{j,t}, 0) = t$ is bounded,
4. Passing to subsequence, $\varphi_{j,t}$ converges strongly to $\varphi_t \in \mathcal{E}^1_0$
5. lower semicontinuity implies
$$t\sigma' \geq \liminf_{j \to +\infty} M_j(\varphi_{j,t}) \geq M(\varphi_t) \geq \sigma \ \mathrm{d}_{1,G}(\varphi_t, 0) - C = t\sigma - C$$
6. it only remains to chose $t > C/(\sigma - \sigma')$. This is a consequence of $M_j \geq -Cd(0, \varphi)$.

□

2. SETUP

functionals

$$M_j \colon \mathcal{E}^{1,T}_j \to \mathbb{R} \cup \{+\infty\}, \quad M \colon \pi^\star \mathcal{E}^{1,T}_X \to \mathbb{R} \cup \{+\infty\}$$

respectively defined by

$$M_j(\varphi) := \mathrm{M}^{\mathrm{rel}}_j(\varphi) = \mathrm{H}_{j,v}(\varphi) + \mathrm{R}_{j,v}(\varphi) + \mathrm{E}_{j,vw\ell_j}(\varphi), \quad \varphi \in \mathcal{E}^{1,T}_j,$$

and

$$M(\pi^\star \psi) := \mathrm{M}^{\mathrm{rel}}_X(\psi) = \mathrm{H}_{X,v}(\psi) + \mathrm{R}_{X,v}(\psi) + \mathrm{E}_{vw\ell_X}(\psi), \quad \psi \in \mathcal{E}^{1,T}_X.$$

$$\mathcal{E}^{1,T}_j := \{\varphi \in \mathrm{PSH}^T(\omega_j) | d_{1,j}(\varphi, 0) < +\infty\}$$

3. ENTROPY GROWTH

Lemma 3.1. *There exists $\delta, C > 0$ such that*

$$\mathrm{M}_j^{\mathrm{rel}}(\varphi) \geq \delta\, \mathrm{H}_j(\varphi) - C\left(\mathrm{d}_{1,j}(\varphi, 0) + 1\right)$$

for all j large enough and $\varphi \in \mathcal{E}_j^{1,T}$.

Lemma 3.2.

$$\mathrm{H}_{j,v}(\varphi) + \mathrm{R}_{j,v}(\varphi) + \mathrm{E}_j(\varphi) \geq \delta\, \mathrm{H}_j(\varphi) - C\left(\mathrm{d}_{1,j}(\varphi, 0) + 1\right)$$

Lemma 3.3.

$$|\mathrm{E}_v(\varphi) - \mathrm{E}_v(\psi)| \leq A(v)\ \mathrm{d}_1(\varphi, \psi) \tag{3.1}$$

(1)

$$\left|\mathrm{E}_{j,vw\xi_j^{\mathrm{ext}}}(\varphi)\right| \leq C\, \mathrm{d}_{1,j}(\varphi, 0).$$

(2) assume $\nu_Y = \widehat{\nu}_Y$ and lemma 4.22 yields

$$\mathrm{R}_{j,v} \geq -C(\mathrm{d}_{1,j}(\varphi, 0) + 1). \tag{3.2}$$

(3) it is enough to show the $(1-\delta)\, \mathrm{H}_{j,v}(\varphi)$ is bounded below.

(4) Two way to show:

$$\mathrm{Ent}(\mu|\nu) = \sup_{g \in C^0(X)} \left\{ \int g\, d\mu - \mu(X) \log \int e^g d\nu \right\} + \mu(X) \log \mu(X), \tag{3.3}$$

$$\mathrm{Ent}(\mu|\nu) \geq \mu(X) \log \frac{\mu(X)}{\nu(X)}. \tag{3.4}$$

(5) Pick a quasi-psh function f on Y such that $\widehat{\nu}_Y = e^{-2f}\nu_Y$ has finite total mass and $\mathrm{Ric}(\widehat{\nu}_Y) = \mathrm{Ric}(\nu_Y) + dd^c f$ is π-semipositive.

(6)

$$\widehat{\mathrm{R}}_{j,v}(\varphi) := -\, \mathrm{E}_{j,v}^{\mathrm{Ric}^T(\widehat{\nu}_Y)}(\varphi), \quad \widehat{\mathrm{R}}_{Y,v}(\pi^\star\psi) := -\, \mathrm{E}_{Y,v}^{\mathrm{Ric}^T(\widehat{\nu}_Y)}(\pi^\star\psi)$$

(7)

$$\widehat{\mathrm{R}}_{j,v}(\varphi) = \mathrm{R}_{j,v}(\varphi) - \int_Y f\, \mathrm{MA}_{j,v}(\varphi) + c_j, \quad \varphi \in \mathcal{E}_j^{1,T}, \tag{3.5}$$

where

$$c_j := \int_Y f\, \mathrm{MA}_{j,v}(0) \to \int_Y f\, \mathrm{MA}_{Y,v}(0) = 0,$$

(8) $\widehat{\mathrm{R}}_{j,v}(\varphi) \geq -C(\mathrm{d}_{1,j}(\varphi, 0) + 1)$ for $\varphi \in \mathcal{E}_j^{1,T}$, where $C > 0$ denotes a uniform constant that is allowed to vary from line to line. Combined with (3.5) this yields

$$\mathrm{R}_{j,v}(\varphi) \geq \int_Y f\, \mathrm{MA}_{j,v}(\varphi) - C_j(\mathrm{d}_{1,j}(\varphi, 0) + 1). \tag{3.6}$$

(9)

$$\widehat{\mathrm{H}}_{j,v}(\varphi) := \tfrac{1}{2}\, \mathrm{Ent}\left(\mathrm{MA}_{j,v}(\varphi) | \widehat{\nu}_Y\right), \mathrm{H}_{j,v}(\varphi) = \widehat{\mathrm{H}}_{j,v}(\varphi) - \int_Y f\, \mathrm{MA}_{j,v}(\varphi)$$

(10)

$$\frac{1}{p}\,\mathrm{H}_{j,v}(\varphi) = \frac{1}{p}\,\mathrm{Ent}(\mathrm{MA}(\varphi)|e^{-f}\nu_Y) = \frac{1}{p}\,\mathrm{Ent}\Big(\mathrm{MA}_{j,v}(\varphi)|e^{-pf}\nu_Y\Big) - \int_Y f\,\mathrm{MA}_{j,v}(\varphi) \geq -C - \int_Y f\,\mathrm{MA}_{j,v}(\varphi) \quad (3.7)$$

(11)

$$\begin{aligned}\mathrm{M}^{\mathrm{rel}}_j(\varphi) &= \mathrm{H}_{j,v}(\varphi) + \mathrm{R}_{j,v}(\varphi) + \mathrm{E}_{j,vw\xi_j^{\mathrm{ext}}}(\varphi)\\ &\geq (1-p^{-1})\,\mathrm{H}_{j,v}(\varphi) - C\big(\,\mathrm{d}_{1,j}(\varphi,0)+1\big).\end{aligned}$$

4. LOWER SEMICONTINUITY PROPERTY $\liminf_j M_j(\varphi_j) \geq M(\pi^\star\psi)$,

Assume strong convergence

Lemma 4.1.

$$\liminf_j \widehat{\mathrm{H}}_{j,v}(\varphi_j) \geq \widehat{\mathrm{H}}_{Y,v}(\pi^\star\psi),$$

$$\mathrm{Ent}(\mu|\nu) = \sup_{g\in C^0(X)}\left\{\int g\,d\mu - \mu(X)\log\int e^g d\nu\right\} + \mu(X)\log\mu(X), \quad (4.1)$$

(1) $\left\{\int g\,d\mu - \mu(X)\log\int e^g d\nu\right\}$ and $\mu(X)\log\mu(X)$ are continous according to μ
(2) is convex and lsc on the space $\mathcal{M}$ of positive Radon measures (equipped with the weak topology)
(3)

$$\mathrm{Ent}(\cdot|\nu)\colon \mathcal{M} \to \mathbb{R}\cup\{+\infty\}$$

Lemma 4.2. *while Lemma 4.22 implies*

$$\liminf_j \widehat{\mathrm{R}}_{j,v}(\varphi_j) \geq \widehat{\mathrm{R}}_{Y,v}(\pi^\star\psi).$$

On the other hand, since $\ell_j \to \ell_X$ smoothly, uniform Lipschitz estimate 1$\Rightarrow$ by Lemma 2.13 strong convergence

$$\lim_j \mathrm{E}_{j,vw\ell_j}(\varphi_j) = \mathrm{E}_{X,vw\ell_X}(\psi).$$