1. Talk on Singularities: 24/04

Def 1: A pair (X, B) consists of a normal variety and a divisor $B = \sum b_i B_i$, $b_i \in [0, 1]$, such that $K_X + B$ is $\mathbb{Q}$-cartier.

i.e $m(K_X + B)$ is Cartier for some $m \in \mathbb{N}$.

Def 2: A log resolution of singularities of a pair (X, B) is a birational morphism $\phi : W \to X$. Such that W is smooth and $\phi^{-1} \operatorname{Supp} B \cup \operatorname{Exc}(\phi)$ is S.N.C.

Here $\operatorname{Exc}(\phi) = \bigcup C$, where $C \subset W$ are curves contracted by ϕ.

(Exist by Hironaka's thm. $/\mathbb{C}$)

Def 3: Let (X, B) be a pair and $\phi : W \to X$ be a log resolution.

Choosing K_W, K_X, so that $\phi_* K_W = K_X$

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(2) We can write

$$K_W + B_W = \phi^*(K_X + B)$$

We say that the singularities of (X,B) are

1) Terminal if each coeff of B_W is ≤ 0, and < 0 for Exceptional components

2) Canonical if each coeff of B_W is ≤ 0.

3) KLT if each coeff of B_W is < 1.

4) LC if each coeff of B_W is ≤ 1.

5) DLT if we can choose ϕ so that the coeff of B_W is < 1 in each exceptional component.

Lemma 4 (Well-known): Def of terminal, canonical, KLT, LC is independent of the choice of the log-resolution.

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Example of non-normal surface(s):

1) $X = \operatorname{Spec} k[x,y,z]/(z^2 - x^2y^2)$
$= \operatorname{Spec} A$

This surface is not irreducible

since $(z-xy)(z+xy) = 0$

A is not integrally closed in its field of fraction

Take $f = \frac{z}{xy}$

$\to f^2 = 1$ But $f \notin A$ because

$\frac{z}{xy}$ is not a polynome

2) take $X = \operatorname{spec} k[x,y,z]/(z^2 - x^2y)$
$= \operatorname{Spec} A$

A is not integrally closed in its field of fraction.

Take $\boxed{f = \frac{z}{x}}$

(4) Examples of singularities:

1) Let (X, B) be a log smooth pair: X is smooth and $\operatorname{Supp} B$ is SNC.

Then (X, B) is

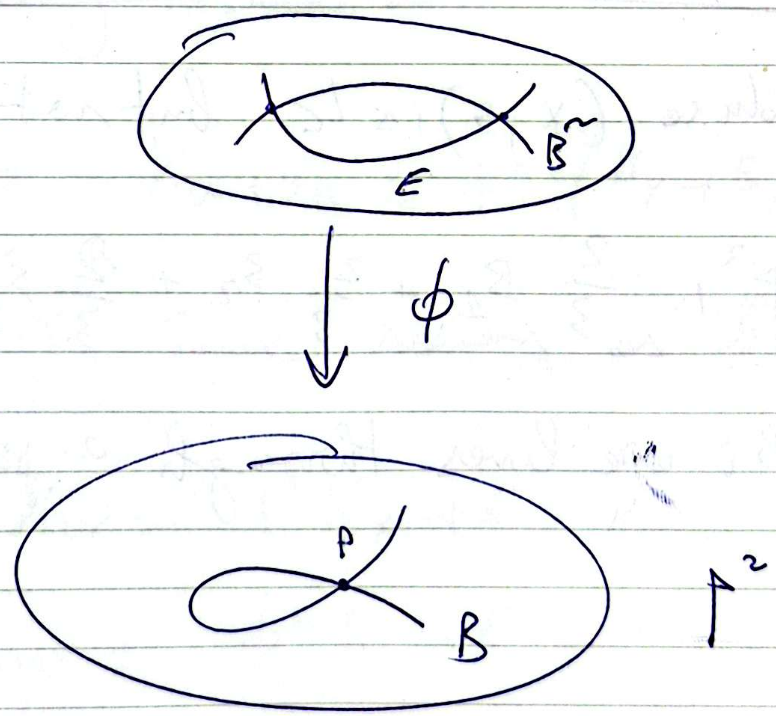
A) terminal if each coeff. of B is 0.

B) canonical if each coeff of B is 0.

C) KLT if each coeff of $B < 1$

etc.

2) Let $X = \mathbb{P}^2$, B nodal cubic curve. Then (X, B) is lc.



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Indeed, $\phi : Bl_p X = W \to X$ is a log resolution.

We have

$$K_W + B_W = \phi^*(K_X + B)$$

But W is just the $Bl_p \mathbb{P}^2 \Rightarrow$

$$K_W = \phi^* K_X + E$$

$$\Rightarrow \quad B_W = \phi^* B - E = \tilde{B}$$

So,

$$\boxed{K_W + \tilde{B} + E = \phi^*(K_X + B)}$$

We deduce (X, B) is lc but not klt.

3) $\left(\mathbb{P}^2, \frac{2}{3} B_1 + \frac{2}{3} B_2 + \frac{2}{3} B_3\right)$

where B_i are lines through a point p.

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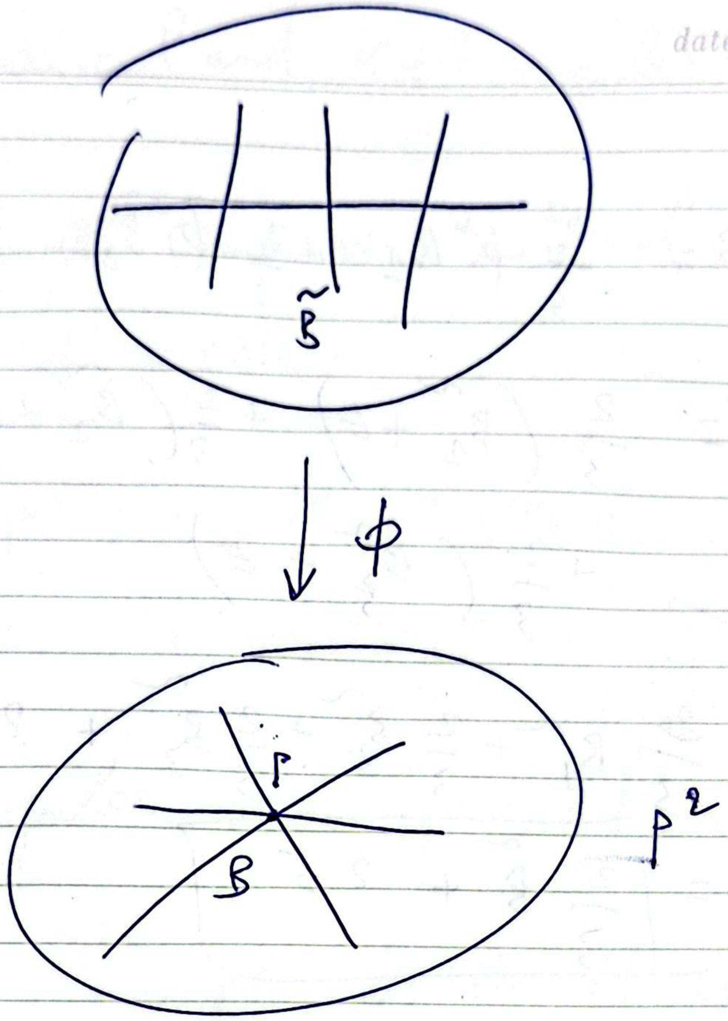

It is obvious that $\phi : W = Bl_p P^2 \longrightarrow X$ is a log resolution.

Write $K_W + B_W = \phi^* (K_X + B)$

For the same reasons as before

$$K_W = \phi^* K_X + E .$$

So $B_W = \phi^* B - E$

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But

$$\phi^* B = \frac{2}{3}\phi^* B_1 + \frac{2}{3}\phi^* B_2 + \frac{2}{3}\phi^* B_3$$

$$= \frac{2}{3}(\tilde{B_1} + E) + \frac{2}{3}(\tilde{B_2} + E) + \frac{2}{3}(\tilde{B_3} + E)$$

$$= \frac{2}{3}\tilde{B_1} + \frac{2}{3}\tilde{B_2} + \frac{2}{3}\tilde{B_3} + 2E$$

$$= \boxed{\frac{2}{3}\tilde{B} + 2E}$$

So

$$\boxed{K_W + \tilde{B} + E = \phi^*(K_X + B)}$$

$\Rightarrow (X, B)$ is Lc

Easy Exercice: $(X, \frac{1}{3}B_1 + \frac{1}{3}B_2 + \frac{1}{3}B_3)$ is klT

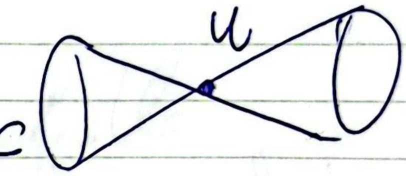
and $(X, B_1 + B_2 + B_3)$ is not lc.

4) Canonical and K-L-T singularities 'Example

take ~~X~~ cone over rational curve C of degree n.

$C \subset P^2$: $x^{n}+y^{n}+z^{n}=0$

$X \subset P^3$: $x^n+y^n+z^n=0$



X is singular on $u=(0,0,0,1)$.

Let $\pi : V \to P^3$ be the blowup at u :

Here ϕ // $V = Bl_u P^3$. , E is the Exceptional divisor $\simeq P^2$.

Let $Y \subseteq V$ be the birational transf of X

and $Y \longrightarrow X$ has one Exceptional curve $C \subseteq E \simeq P^2$. given by the equation : $x^n+y^n+z^n=0$ on P^2.

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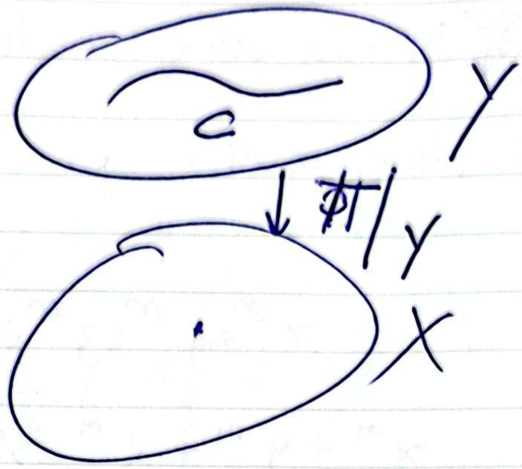
we have this picture



take H: hyperplane passing through l. Then

$$\pi^* H = G + E$$

G : strict transform.

Let $L = G|_E$. We see that

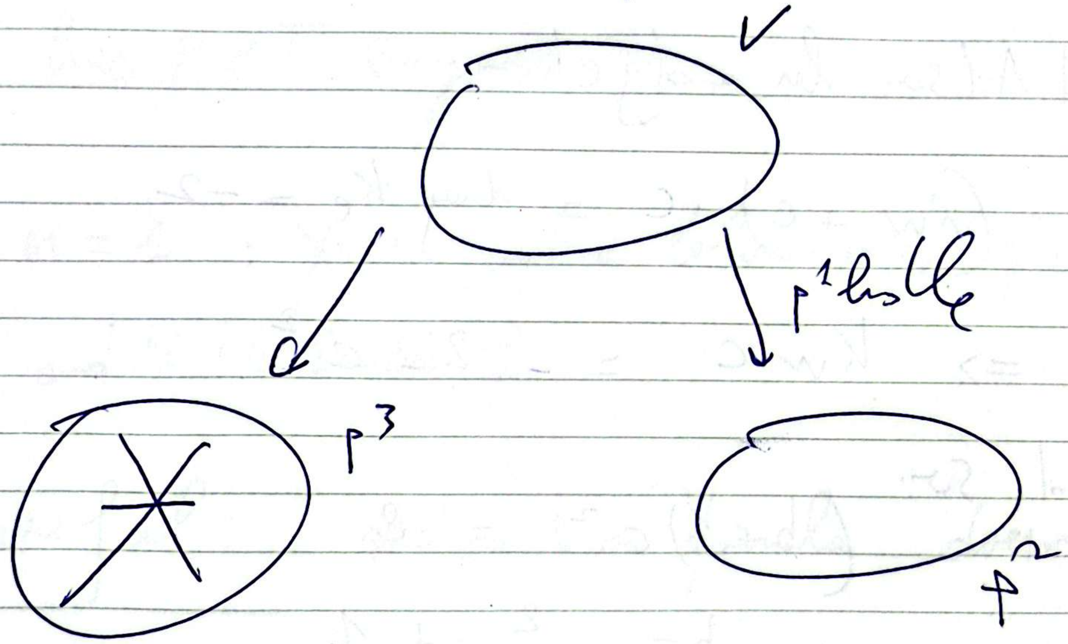
$$L + E|_E \sim 0.$$

and L is a line on $E \simeq \mathbb{P}^2$.

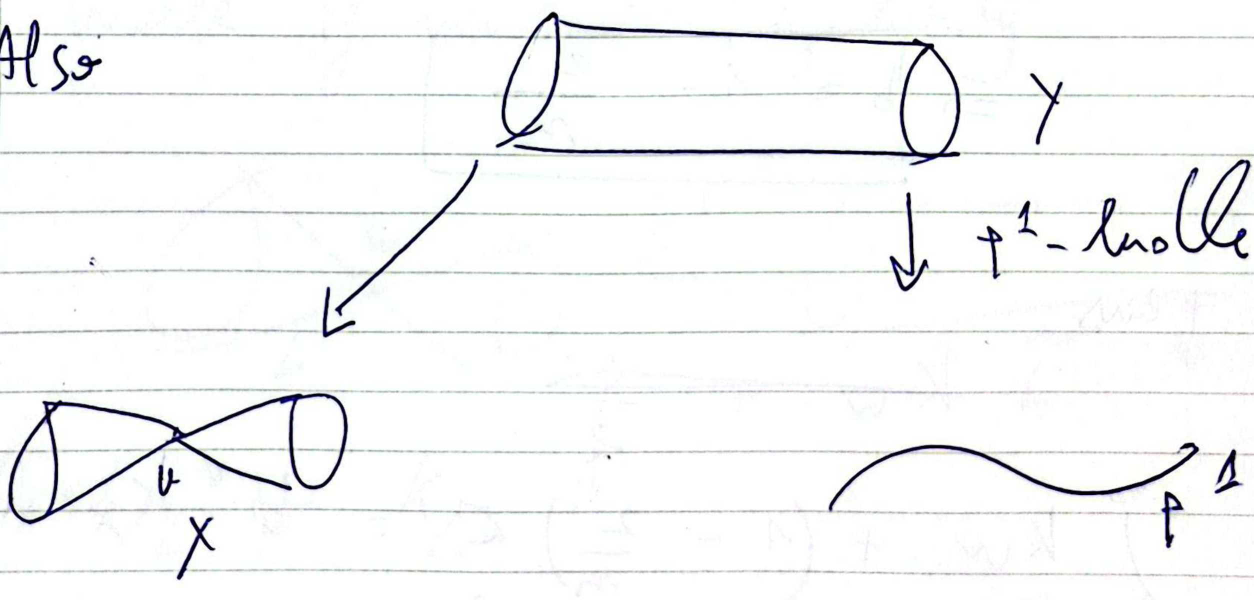
Now, we compute C^2

$$C^2 = (C.C)_Y = (E|_Y . C)_Y$$
$$= (E.C)_V$$
$$= (E|_E)_V = -2 \cdot C = \boxed{-n}$$

Remark that V is P^1-bundle



Also



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Now, Let's see what the singularity.

$$K_W + B_W = \phi^* K_X$$

$$B_W = bE$$

So we have

$$(K_W + bC)C = \phi^* K_X \cdot C = 0$$

and Also lu adjuction.

$$(K_W + C) \cdot C = \deg K_C = -2$$

$$\Rightarrow K_W C = -2 - C^2$$

and so.

$$(b-1)C^2 = 2$$

$$\Rightarrow b = \frac{2}{C^2} + 1$$

$$\Rightarrow \boxed{b = 1 - \frac{2}{m}}$$

Thus

$$\cancel{K_W + \frac{1}{2}}$$

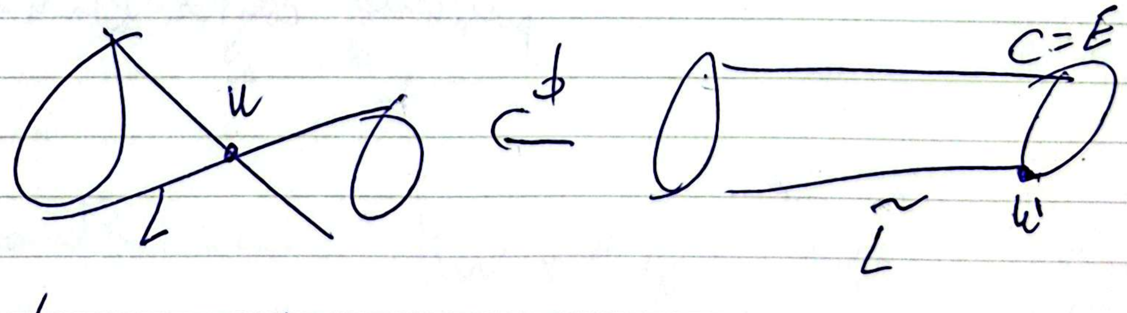
$$\begin{cases} K_W + \left(1 - \frac{2}{m}\right) C = \phi^* K_X \\ K_W \cdot C = -2 - C^2 \end{cases}$$

if $n=2$: X is a cone over a conic curve and X is canonical with C^2 is (-2)-curve.

if $n \geq 3$: X is a cone over a rational curve of degree n, and X has klT singularity.

if $n=1$: $\tilde{X}$ is just a blow-up of X and X is smooth.

<u>Be careful:</u> Adjunction formula doesn't work if we have singularity.



$K_X+L/_L \neq K_L$:

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$$K_X + L|_L = \phi^*(K_X + L)|_{\tilde{L}}$$

$$= \left(K_Y + \left(1 - \frac{2}{n}\right)C + \tilde{L} + \frac{1}{n}C\right)\Big|_{\tilde{L}}$$

(because $\phi^* L = \tilde{L} + \frac{1}{n}C$

Check: $(\tilde{L} + \frac{1}{n}C) \cdot C = 0$)

$$= K_Y + \tilde{L}|_{\tilde{L}} + \left(1 - \frac{1}{n}C\right)\Big|_{\tilde{L}}$$

$$= K_L + \left(1 - \frac{1}{n}\right)u$$

$$= \boxed{K_L + \left(1 - \frac{1}{n}\right)u}$$

positive contribution.

Example of terminal singularity: (Threefold)

Let $X \subseteq \mathbb{P}^4 : xy - zu = 0$

This is a cone over a smooth quadric surface $Q: xy - zu = 0 \subset \mathbb{P}^3$.

X has $u = (0,0,0,0,1)$ as singularity.

$\Phi : W = Bl_u X \longrightarrow X$ is a resolution.

$$K_W = \phi^* K_X + ?$$

$$\boxed{K_W + B_W = \phi^* K_X}$$

$$\begin{tikzcd} & E \arrow[dl, "f_1"'] \arrow[d, "f_2"] \\ \mathbb{P}^1 & \mathbb{P}^1 \end{tikzcd}$$

By adjunction: $K_W + E|_E = K_E$

$\boxed{\text{Claim: } E|_E \simeq \mathcal{O}_E(-1,-1)} \Rightarrow K_W . E = \mathcal{O}_E(-1,-1)$

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proof of Claim:

$$Y = Bl_p\,\mathbb{P}^4$$

$$\mathcal{O}_W(E) = \mathcal{O}_Y(H)\big|_W$$

and so

$$\mathcal{O}_E(E) = \mathcal{O}_W(E)\big|_E = \mathcal{O}_Y(H)\big|_E$$

$$= \mathcal{O}_E(-1)$$

it is not hard to see that

E is $(-1,-1)$ curve..

we have

$$K_W + bE = \phi^* K_X$$

$$\Rightarrow (K_W + bE)\cdot L = 0$$

$$\Rightarrow \boxed{b = -1}$$

$$\Rightarrow K_W = \phi^* K_X + E$$

$\Rightarrow$ $(X, 0)$ is terminal.

This is an Example of terminal singularity that is not smooth.

Note that terminal singularity in dimension 2 are smooth.

Lemma 5: (Negativity lemma):

Let $\varphi : Y \longrightarrow X$ be a projective birational morphism of normal varieties. Let Δ be $\mathbb{Q}$-Cartier such that

1) $-\Delta$ is nef over X.

2) $\varphi_* \Delta \geqslant 0$

$\Longrightarrow$ $\boxed{\Delta \geqslant 0}$

$\boxed{\dim X = 2}$

Lemma 6: Let $\phi : Y \longrightarrow X$ be a minimal resolution of X. Assume that K_X is $\mathbb{Q}$-Cartier and write

$$K_Y + B_Y = \phi^* K_X \Rightarrow B_Y \geqslant 0.$$

proof: we have

$$B_Y = -K_{Y/X} \iff$$

$\forall C \subset Y$ such that $\varphi_* C = p$

$$\Rightarrow B_Y \cdot C = -K_Y \cdot C \leq 0$$

Thus by negativity Lemma:

$$B_Y \geq 0$$

Corollary 7: if $\dim X = 2$, terminal singularity $\iff$ smooth.

proof: if X has terminal singularity, Let $\varphi: Y \to X$ be minimal resolution.

$$\Rightarrow B_Y \geq 0 \text{ by above.}$$

But X has terminal singularity $\Rightarrow$

$$B_Y < 0$$

contradiction. (there is no EXC)

~~$B_Y = 0$~~ $\Rightarrow \varphi$ is an isomorphism

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<u>Lemma</u>: if $\dim X = 2$, X is klT.

$\Rightarrow$ Each Exceptional curve of $W \to X$ is isomorphic to $\mathbb{P}^1$.

<u>In Higher dimension:</u> (Shokurov Conj)

+ <u>Hacon-McKernan thm:</u>

Exc set is $\cup$ of CRC subvarieties.

Canonical singularities in $\dim = 2$ are classified which are Du-Val singularities : ADE

A: $x^2 + y^2 + z^{n+1} = 0$

(if $n = 1$, cone over conic)

D: $x^2 + y^2 z + z^{n-1} = 0$

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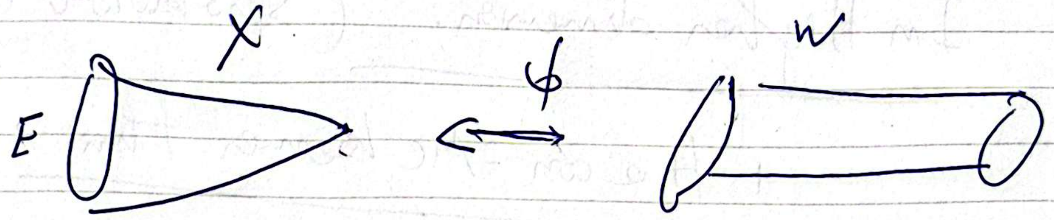
it is well know also that surface K3T are quotient.

Ex for LC: cone over elliptic curve



$K_W = \phi^* K_X + ?$

or $K_W + B_W = \phi^* K_X$

$$\boxed{B_W = bE}$$

By adjunction:

$K_W + E|_E = 0$

and $K_W + bE|_E = 0$

$(b-1)E^2 = 0$

since $E^2 < 0 \Rightarrow \boxed{b = 1}$

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$$\Rightarrow K_W + E = \phi^* K_X$$

X is Lc.

Ex for non Lc:

cone over curve of higher genus $g \geq 2$.

(Easy exercice).

Q-factorial singularities:

Def 3: Le X be normal variety,

D is $\mathbb{Q}$-cartier if mD is Cartier.

$\mathbb{Q}$-factorial if every Weil divisor is $\mathbb{Q}$-Cartier.

Ex: klT surface $\Rightarrow$ quotient $\Rightarrow$ $\mathbb{Q}$ factorial.

Counter Ex for Lc and not Q-factorial:

Take X is a cone over an elliptic curve $E \subseteq \mathbb{P}^2$. by a cubic equation. $(y^2 z = x^3 + a x z^2 + b z^3)$

X is singular on. $u = (0,0,0,1)$

take blow-up on $(0,0,0,1)$

$$\phi : V = Bl_u X \longrightarrow X.$$

A well known fact is that $\exists$ $\mathbb{P}^1$-bundle

$$f : W \longrightarrow E.$$

$$\begin{tikzcd}
C \arrow[r] \arrow[dr, "\simeq"'] & W \arrow[d, "f"] \arrow[dr, "\phi"] & \\
& E & X
\end{tikzcd}$$

Now, pick a divisor L on E

such that $L \equiv 0$ but

$$mL \not\equiv 0$$

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Let $D = \phi_* f^* L$. Then

D is not $\mathbb{Q}$-Cartier..

<u>proof:</u> If mD is Cartier.

Then $\phi^* mD = mf^*L$ by

applying the negativity lemma

to $\phi^* mD - mf^*L$ and

$mf^*L - \phi^* mD$

Therefore,

$$mf^*L|_C = \phi^* mD|_C \sim 0$$

$\Rightarrow mL \sim 0$

which is a contradiction.

In higher dimension:

terminal singularity $\not\Longrightarrow$ $\mathbb{Q}$-factorial.

$X \subset \mathbb{P}^4 \quad : \quad xy - zu = 0$

X is a cone over a quadric surface

$\phi : W \longrightarrow X$

ϕ is a resolution.

$$\begin{tikzcd}
 & W \arrow[dl] \arrow[dd, "\phi"] \arrow[dr] & \\
Y \arrow[dr, "f"'] & & Y' \arrow[dl, "f'"] \\
 & X &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[dl] \arrow[dd] \arrow[dr] & \\
\mathbb{P}^1 \arrow[dr] & & \mathbb{P}^1 \arrow[dl] \\
 & pt &
\end{tikzcd}$$

pick H on Y such that

$H \cap C \neq 0$

C is the exceptional curve

of $Y \to X$ but $C \notin H$.

take $D = f_* H$. Then D is not $\mathbb{Q}$-Cartier: if mD is Cartier,

then $(f^* mD) \cdot C = 0$ but

$f^* mD = mH$ as f doesn't contract any divisor, while

$mH \cdot C > 0$

contradiction.

Quotient singularities:

Let X be a normal variety. Let

$\mathrm{Aut}(X) = \{ g : X \to X,\ g \text{ isomorphism} \}$.

take $G \leq \mathrm{Aut}(X)$ is a finite subgroup.

Well known fact $Y : X/G$ is normal

$\pi : X \to Y$ is quotient finite map

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<u>Ex</u>: Let $X = A^2$ and

$$\sigma : X \longrightarrow X$$
$$(a, b) \longrightarrow (-a, -b)$$

Let $G = \langle \sigma \rangle \leq Aut(X)$.

$\Rightarrow |G| = 2$. Let $Y = X/G$.

writing $X = Spec\, K[\alpha, \beta]$

and $Y = Spec\, k[\alpha, \beta]^G$

where

$$K[\alpha, \beta]^G = \{ f \in K[\alpha, \beta],\ f \text{ is } G\text{-invariant} \}$$

Note that G acts on $K[\alpha, \beta]$ by

$$\alpha \longrightarrow -\alpha$$
$$\beta \longrightarrow -\beta$$

easy calculation: $K[\alpha, \beta]^G = K[\alpha^2, \alpha\beta, \beta^2]$

So $Y = Spec\, K[\alpha^2, \alpha\beta, \beta^2]$

~~$\Rightarrow X = Sp$~~

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Define

$$\phi : k[s,t,u] \to k[\alpha^2, \alpha\beta, \beta^2]$$

by $s \to \alpha^2$, $t \to \alpha\beta$

$u \to \beta^2$

Then we can check:

$$\ker\phi = \langle su - t^2 \rangle$$

So $k[\alpha^2, \alpha\beta, \beta^2] \simeq k[s,t,u]/\langle su - t^2 \rangle$

$\Rightarrow y$ is cone over a conic.