Goal: Weighted Poincaré and Sobolev inequalities on certain complete non compact mfds with polynomial volume growth and lower Ricci bound.

Def: (M^n, g) complete non compact, $n > 2$, $\beta \in \mathbb{R}^+$. M is called $SOB(\beta)$ if $\exists\, x_0 \in M$, $C \geq 1$ s.t:

- $A(x_0, s, t)$ is connected for all $t > s \geq C$ (Only one end)
- $|B(x_0, s)| \leq C s^\beta, \quad \forall s \geq C$

- $\left| B\left(x, \left(1-\frac{1}{c}\right) r(x)\right) \right| \geq \frac{1}{c}\, r(x)^{\beta}$

$\forall\ r(x) := \mathrm{dist}(x_0, x) \geq C$

- $\mathrm{Ric}(x) \geq -C\, r(x)^{-2}$

$\forall\ r(x) \geq C$.

Example: $\exists$ if, outside a compact subset,

M is quasi-isometric to

$C(L) \times N$ (or $\mathbb{R}^+ \times N$)

where $C(L)$ is a Riemannian cone

and N is closed with $\mathrm{Ric} \geq 0$

then M is $SOB(\beta)$ with $\beta = \dim C(L)$ (or $\beta = 1$)

Main Theorem: Suppose (M, g) is $SOB(\beta)$ for some $\beta \in \mathbb{R}^+$.

(i) $\forall \varepsilon > 0$, there exists a positive step function $\psi_\varepsilon \sim (1+r)^{-\max\{\beta, 2\} - \varepsilon}$ on M, s.t, $\forall \alpha \in \left[1, \frac{n}{n-2}\right]$, $u \in C_0^\infty(M)$,

$$\left(\int_M |u - u_\varepsilon|^{2\alpha} \cdot (1+r)^{\alpha(\min\{\beta-2, 0\} - \varepsilon) - \beta} \, d\mathrm{vol} \right)^{\frac{1}{2\alpha}} \leq C(\varepsilon) \|\nabla u\|_{L^2}$$

u_ε : is the average of u w.r.t $\Psi_\varepsilon\, d\mathrm{vol}$

(ii) If $\beta > 2$, $\forall \alpha \in \left[1, \frac{n}{n-2}\right]$, $u \in C_0^\infty$,

$$\left(\int_M |u|^{2\alpha} \cdot (1+r)^{\alpha(\beta-2)-\beta}\, d\mathrm{vol}\right)^{\frac{1}{2\alpha}} \leq C \|\nabla u\|_{L^2}$$

This, in particular, applies to the examples above. The weights are sharp on model spaces such as $\mathbb{R}^\beta \times T^{n-\beta}$, with $\beta = 1, 2, \dots, n$.

The proof will be divided into

2 main steps:

① A Sobolev inequality with Dirichlet boundary conditions on Balls and annuli

② A patching argument.

I - Dirichlet-type Sobolev inequalities on Balls and annuli:

Notation:

Notation: Let (M,g) be a Riemannian manifold (not necessarily complete). $x_0 \in M$.

- $B(r) := B(x_0, r)$.
- $A(r_1, r_2) := A(x_0, r_1, r_2)$.

- If $\Omega_0 \subseteq M$ is open and precompact, we denote $\forall p \geq 1,\ \alpha \geq 1$

$$\boxed{DS(\Omega_0, p, \alpha) := \sup_{u \in} \frac{\|u\|_{\alpha p}}{\|\nabla u\|_{p}}}$$

$\underset{\substack{C_0^\infty(\Omega_0) \\ u \neq 0}}{} \ \| \ \ \|_p$

By Hölder's inequality:

$$DS(\Omega_0, p, \alpha) \leq \frac{\alpha p}{\alpha'} DS(\Omega_0, 1, \alpha')$$

$$\text{if } \quad 1 - \frac{1}{\alpha'} = \frac{1}{p}\left(1 - \frac{1}{\alpha}\right)$$

Proposition (Dirichlet-type Sobolev ineq.):

(i) If $2\sigma s < \mathrm{diam}(M)$,

$\overline{B(2\sigma s)}$ compact, $\mathrm{Ric} \geq -\Lambda s^{-2}$

in $B(2\sigma s)$, $\Lambda \geq 0$, then on $B(20s)$, -1 //

$\forall p \in [1, n)$ and $\alpha \in \left[1, \frac{n}{n-p}\right]$,

$$DS(B(s), p, \alpha) \leq C(n, p, \Lambda) \cdot s \cdot |B(s)|^{\frac{1}{p}\left(\frac{1}{\alpha}-1\right)}$$

ii If $20s < \min\{r, \operatorname{diam}(M)\}$

$\overline{B(r+20s)}$ compact,

$Ric \geq -\Lambda s^{-2}$ on $A(r-20s, r+20s)$,

$\Lambda \geq 0$, $\forall p \in [1, n)$, $\alpha \in \left[1, \frac{n}{n-p}\right]$,

$$DS(A(r, r+s), p, \alpha) \leq C(n, p, \Lambda)\, s \cdot \sup_{x \in A(r, r+s)} |B(x, s)|^{\frac{1}{p}\left(\frac{1}{\alpha}-1\right)}$$

Lemma 1 (Peter Li): For all $\Omega_0 \subseteq M$ open and precompact:

$$DS(\Omega_0, 1, \alpha) = \sup \left\{ \frac{|\Omega|^{\frac{1}{\alpha}}}{|\partial\Omega|}, \ \Omega \subseteq \Omega_0, \ \partial\Omega \text{ smooth} \right\}$$

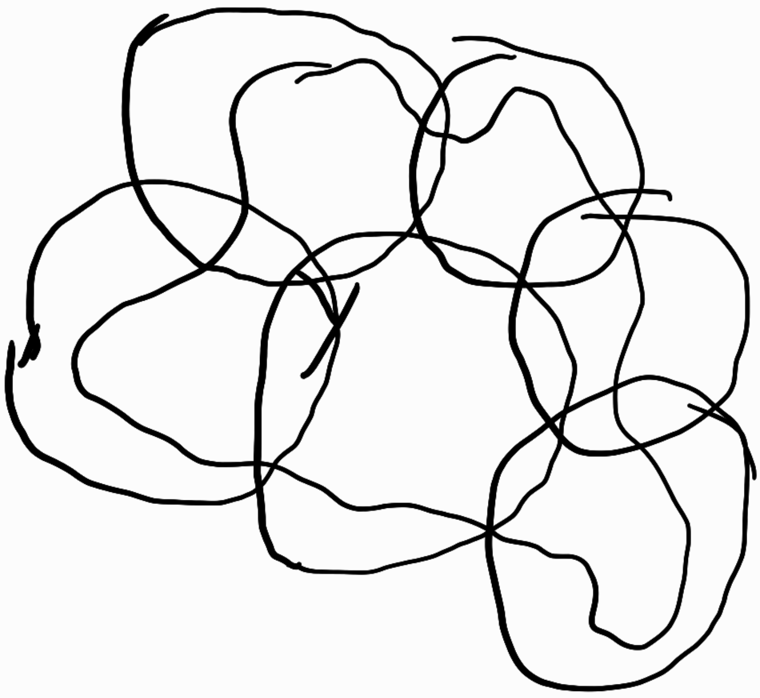
$\longrightarrow$ We need an isoperimetric inequality!

Def: X metric measure space. $Y \subset X$. $\mathcal{B}$ a covering of Y by balls is called

By Be…

(ε, r_0)-good if:

- $\mathrm{Center}(B) \in Y, \ \forall B \in \mathcal{B}$
- $\mathrm{radius}(B) \leq r_0$
- $\min\{|B \setminus Y|, |B \cap Y|\} \geq \varepsilon |B|$



Lemma 2: $\Omega \subseteq M$ with smooth $\partial\Omega$,

$\mathcal{B}$ an (ε, r_0)-good covering of

$\mathcal{B}$ an (εr_0)-good covering of $|\Omega|$. Let N be the $5r_0$-neighbourhood of Ω and assume $\overline{N}$ is compact. If $\mathrm{Ric} \geq -(n-1)\delta^2$ on N with $0 \leq r_0 \delta \leq \Lambda$ and if $\alpha \geq 1$, then

$$\boxed{\frac{|\Omega|^{\frac{1}{\alpha}}}{|\partial\Omega|} \leq C(n,\varepsilon,\Lambda) \sup_{B \in \mathcal{B}} |B|^{\frac{1}{\alpha}-1} \operatorname{radius}(B)}$$

Lemma 3:

(i) For all $\Lambda > 0$, $\exists\, \varepsilon(n, \Lambda) > 0$ such that if $s < \operatorname{diam}(M)$,

$B := B(s)$, $\overline{9B}$ is compact,

$\operatorname{Ric} \geq -\Lambda s^{-2}$ on $9B$, then

$\forall\, \Omega \subset B$, $\forall\, x \in \Omega$, there exists $0 < r_{x,\Omega} \leq 4s$ s.t

$$\left| B(x, r_{x,\Omega}) \setminus \Omega \right| = \varepsilon \left| B(x, r_{x,\Omega}) \right|$$

(ii) $\forall\, \Lambda, N > 0$, $\exists\, \varepsilon(\Lambda, N, n) > 0$

s.t. if $r > 6t$, $s \leq Nt$,

$A := A(r, r+s)$, $\overline{B(r+2s+2t)}$ is compact, $\mathrm{Ric} \geq -\Lambda t^{-2}$ on

$A(r-6t, r+2s+2t)$, $\forall x \in \Omega \subset A$

$\exists \quad 0 < r_{x,\Omega} \leq r_x^* := \mathrm{dist}(x_0, x) - r + 2t$

$\leq s + 2t$

with $\boxed{|B(x, r_{x,\Omega}) \setminus \Omega| = \varepsilon |B(x, r_{x,\Omega})|}$

$\Rightarrow$ In particular

(i) + (ii) $\Rightarrow \{(B(x, r_{x,\Omega})), x \in \Omega\}$ is

(ε, r_0)-good for B and A

... with ε as above

respectively ...
and $r_0 = 4s$, $r_0 = s+2t$ respectively

$\Rightarrow$ Proposition is a corollary of Lemma 1, 2 and 3.

Proof of Lemma 2:

First, choose finitely many balls $B_i \in \mathcal{B}$ s.t. the $2B_i$ are pairwise disjoint and the $5B_i$ still cover Ω. (This is possible: Vitali-type selection: Take

B_1 with maximal radius,

B_{i+1} " " " among

those s.t $2B \cap 2B_i = \emptyset$.)

Key estimate:

$$|B_i| \le C(n, \varepsilon, 1)\, r_i\, |\partial\Omega \cap 2B_i|$$

$\Longrightarrow$

$$|\Omega|^{\frac{1}{\alpha}} \le \sum |5B_i|^{\frac{1}{\alpha}} \overset{\text{Bishop-Gromov}}{\le} C(n,1) \sum |B_i|^{\frac{1}{\alpha}}$$

$$|\Omega| \leq \sum |\Omega \cap B_i| \leq C(n,\Lambda) \sum |B_i|$$

$$\leq C(n,\Lambda,\varepsilon) \cdot \sum |B_i|^{\frac{1}{\alpha}-1} r_i |\partial\Omega \cap 2B_i|$$

$$\leq C(n,\Lambda,\varepsilon) |\partial_+\Omega| \sup |B|^{\frac{1}{\alpha}-1} \operatorname{radius}(B)$$

Recall (Bishop-Gromov):

M complete $\mathrm{Ric} \geq (n-1)K$, $p \in M$.

$$\Longrightarrow \phi(r) = \frac{\mathrm{Vol}(B(p,r))}{\mathrm{Vol}_{M_K}(B(p_K,r))} \searrow$$

with $\lim_{r\to 0} = 1$

Proof of Key estimate:

We will prove that

$$\min\{|B_i\setminus\Omega|, |B_i\cap\Omega|\} \leq 2\frac{v_s(4r_i)}{a_s(2r_i)}|\partial\Omega\cap B_i|$$

which implies the estimate by (ε, r_0)-goodness.

The main tool is the Bishop-Gromov volume comparaison (infinitesimal form).

Let $X_1 = \{ (z,z') \in (B_1 \cap \Omega) \times (B_1 \setminus \Omega)$

$\exists !$ minimizing geod. $\gamma : z \longrightarrow z'$

crossing $\partial\Omega$ transversally

and y is the first crossing

and $\operatorname{dist}(z,y) \geq \operatorname{dist}(z',y) \}$

and X_2 the same with the opposite inequality.

$X_1 \cup X_2$ has full-measure.

- Suppose $|X_1| \geq \frac{1}{2} |X_1 \cup X_2|$.

By Fubini, $\exists\, z \in B_i \cap \Omega$ s.t

$$Z' := \{ z' \in B_i \setminus \Omega,\ (z, z') \in X_1 \}$$

has at least half measure of $|B_i \setminus \Omega|$.

We project Σ into

$$\Sigma^{first} \subseteq \partial\Omega \cap 2B_i$$

(first crossing).

For $y \in \Sigma^{first}$, $\gamma_y : z \longrightarrow y$

$$\gamma_y = \exp_z(t\, v_y).$$

Define

$$d_1, d_2 : \Sigma^{first} \longrightarrow \mathbb{R}_+$$

$$\begin{cases} d_1(y) = \operatorname{dist}(z, y) \end{cases}$$

$$d_2(y) = \min\Big\{ \sup\{t>0,\ \gamma_y(d_1(y)+t)\in Z\},$$
$$\sup\{t>0,\ \gamma_y \text{ minimal on } [0, d_1(y)+t]\}\Big\}$$

$\Rightarrow$ γ_y is minimal on $[0, d_1(y)+d_2(y))$

and $\gamma_y([d_1(y), d_1+d_2(y))) \subset Z'$

$\Rightarrow$

$$\mathcal{U} := \{(y,t)\in \Sigma^{\text{first}}\times\mathbb{R}^+,\ d_1(y) < t < d_1(y)+d_2(y)\}$$

$\phi : \mathcal{U} \longrightarrow M$ embedding

$$\phi(y,t) = \gamma_y(t) = \exp_z(v_y t)$$

By standard calculations:

$$\boxed{\phi^*(dvol_M)\Big|_{(y,t)} = \frac{\mathcal{J}(t v_y)}{\mathcal{J}(d_1(y) v_y)} \cos\alpha_y \, darea_{\partial\Omega} \wedge dt}$$

α_y: angle between $\dot{\gamma}_y(d_1(y))$ and the unit normal to $\partial\Omega$ at y

$$\mathcal{J}(w) := |w|^{n-1} \det d\exp_z\Big|_w$$

$$\left(\exp_z^*(dvol_M) = \mathcal{J}(tv)\, dt \wedge dvol_{S_zM}(v) \right)$$

$\Longrightarrow$

$$|Z'| = |\phi(U)|$$

$$\leq \int \int_{d_1}^{d_1+d_2(y)} \frac{\mathcal{J}(t v_y)}{\ldots} dt \, darea(y)$$

$$\int_{\Sigma^{\text{first}}} \int_{d_1(y)} \mathcal{T}(d_1(y)\, v_y)$$

B-G

$$\leqslant \int_{\Sigma^{\text{first}}} \int_{d_1(y)}^{d_1(y)+d_2(y)} \frac{a_s(t)}{a_s(d_1(y))}\, dt\, d\text{area}(y)$$

$$= \int_{\Sigma^{\text{first}}} \frac{v_s(d_1+d_2) - v_s(d_1)}{a_s(d_1)}\, d\text{area}$$

$$\leqslant \int_{\Sigma^{\text{first}}} \frac{v_s(2d_1)}{a_s(d_1)} \leqslant \int \frac{v_s(4r_i)}{a_s(2r_i)}$$

$$\leqslant \frac{v_s(4r_i)}{a_s(2r_i)} |\partial\Omega \cap 2B_i|$$

$\square$

Preuve de Lemme 3:

(i) $x \in B$, Let $x'' \in B(x, 4s)$

$\gamma : x \longrightarrow x''$ and

$x' = \gamma(3s)$.

$$|B(x,4s) \setminus B| \geq |B(x',s)|$$

$$\geq \tilde{\varepsilon}\, |B(x,s)|$$

Volume-comparaison

$$\geq \varepsilon\, |B(x,4s)|$$

but $|B(x,r) \setminus B| = 0$

for r very small,

$\Rightarrow$ by continuity, $\exists\, r_x \in (0, 4s]$

s.t $|B(x, r_x) \setminus B| = \varepsilon |B(x, r_x)|$

(ii) similar with minor technical differences

$\square$