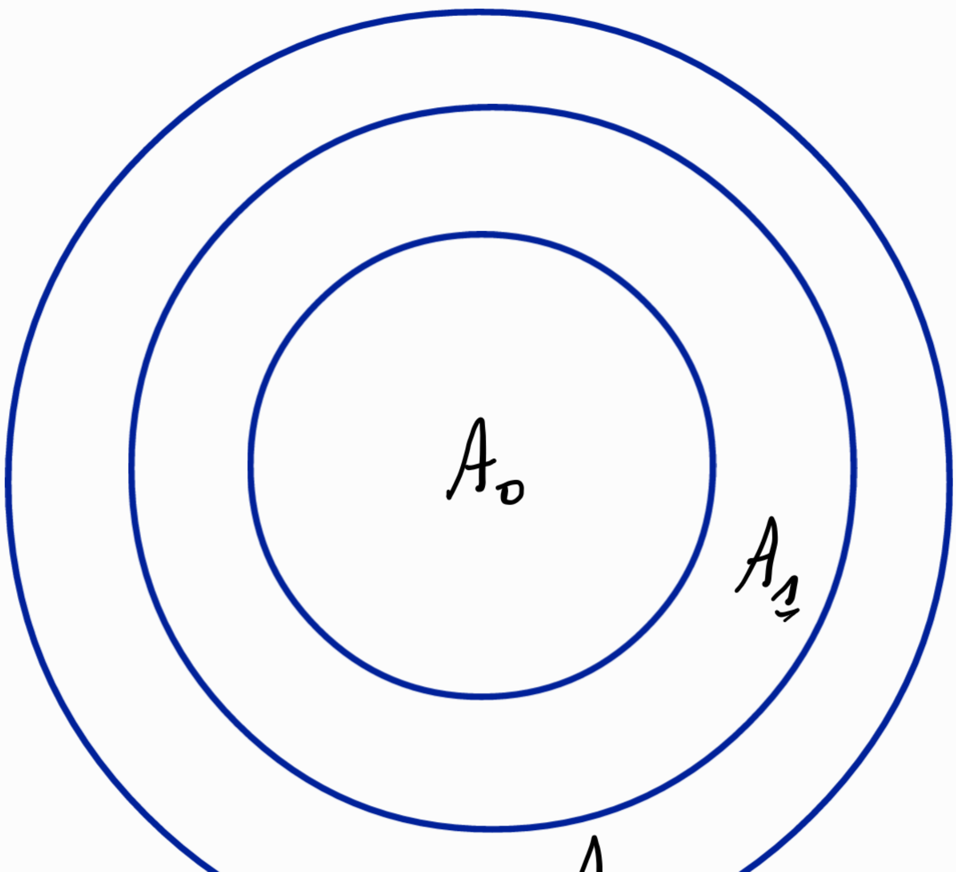
The last time, we established a Dirichlet-type Sobolev inequality on Balls and annuli. We want to use this in the following way to prove the global Sobolev inequalities:

$$A_0 = B(x_0, C_0) \qquad \eta := 1 + \frac{1}{C_0}$$

$$A_k = A(x_0, r_{k-1}, r_k) \qquad r_k = \eta^k C_0$$



Notation:

μ: cst TBD.

φ: fixed "weight" function TBD

$\|\cdot\|_{Y,\varphi,2\alpha}$: the $L^{2\alpha}$-norm w.r.t $\varphi\, dvol$ on Y.

$$\|u-\mu\|^2_{M,\varphi,2\alpha} = \sum_i \|u-\mu\|^2_{A_i,\varphi,2\alpha}$$

$$\leq 2\sum_i \left(\|u-u_i\|^2_{A_i,\varphi,2\alpha} + \|u_i-\mu\|^2_{A_i,\varphi,2\alpha}\right)$$

$$\leq \underbrace{2\sum_i \|u-u_i\|^2_{A_i,\varphi,2\alpha}}_{2} + \underbrace{2\sum_i (\varphi_i |A_i|)^{\frac{1}{\alpha}} |u_i-\mu|^2}_{1}$$

Where:

$$\boxed{u_i := \fint_{\eta A_i} u \quad \text{and} \quad \varphi_i = \sup_{A_i} \varphi}$$

Therefore, we need:

1. A discrete version of a Weighted Neumann-type poincaré inequality.

2. A Neumann-type Sobolev inequality on Balls and annuli

1. Discrete Neumann-type poincaré inequality:

Consider $G=(V,E)$ a (countable) connected graph. For $x,y \in V$ we denote $m_{x,y} = \ell(\gamma^{\{x,y\}}) = \min_{\tilde{\gamma}} \ell(\tilde{\gamma}^{\{x,y\}})$

If $u : V \longrightarrow \mathbb{C}$, we denote

$$u_x = u(x), \quad |\nabla u|_x^2 := \sum_{\substack{y \in V \\ \{x,y\} \in E}} |u_x - u_y|^2$$

Lemma 1: Let $w : V \longrightarrow \mathbb{R}^+$ s.t

$\sum w_x = 1$. Introduce $\overline{w} : V \to \mathbb{R}^+$

by setting

$$\overline{w}_x := \sum_{\substack{(z,z') \in V^2 \\ x \in \gamma^{\{z,z'\}}}} m_{z,z'} \, w_z w_{z'}.$$

Then, for all $u : V \longrightarrow \mathbb{C}$:

$$\sum_{x \in V} w_x u_x = 0 \Longrightarrow \sum_{x \in V} w_x |u_x|^2 \leq \sum_{x \in V} \overline{w}_x |\nabla u|_x^2$$

② Neumann-type Poincaré and Sobolev inequalities:

Lemma 2: For $k \geq 0$, $r > 0$, Let

$B_k := B(x_0, kC_0)$, $A_k := A(x_0, r, \eta k r)$.

Therefore:

$$\boxed{\|u - u_{\eta B_k}\|_{\eta B_k, 2} \leq C(k) \|\nabla u\|^2_{\eta^2 B_k, 2}}$$

$$\boxed{\|u - u_{\eta A_k}\|_{\eta A_k, 2} \leq C(k) \|\nabla u\|^2_{\eta^2 A_k, 2}}$$

Proof: This is a consequence of The Cheeger–Colding Segment inequality, Lemma 1 and the SOB(β) condition.

Recall (Segment inequality):

Let $B(x,r) \subset (M,g)$. If $\overline{2B}$ is compact, $Ric \geq -\Lambda r^{-2}$ on $2B$, $\forall u \in C^\infty(B)$:

$$\int_B |u-u_B|^2 \leq C(n,\Lambda) r^2 \int_{2B} |\nabla u|^2$$

$\square$

Corollary:

$$\|u - u_{\eta B}\|_{B,2\alpha} \leqslant C \, \|\nabla u\|_{\eta^2 B, 2}$$

$$\|u - u_{\eta A}\|_{A,2\alpha} \leqslant C \, r^{1+\frac{\beta}{2}\left(\frac{1}{\alpha}-1\right)} \|\nabla u\|_{\eta^2 A, 2}$$

Proof: $\chi_A \in C_0^\infty(\eta A)$, $\chi_A \equiv 1$ on A, $|\nabla \chi_A| \leqslant C r^{-1}$

$$\|u - u_{\eta A}\|_{A,2\alpha} \leqslant \|\chi_A (u - u_{\eta A})\|_{\eta A, 2\alpha}$$

D-S + SOB(β)

$$\leqslant C \cdot r \cdot r^{\frac{\beta}{2}\left(\frac{1}{\alpha}-1\right)} \cdot \|\nabla(\chi_A (u - u_{\eta A}))\|$$

$$\leqslant C r^{1+\frac{\beta}{2}\left(\frac{1}{\alpha}-1\right)} \left(\|\chi_A \nabla u\| + \|\nabla \chi_A (u - u_{\eta A})\| \right)$$

Lemma 2

$$\leq C r^{1+\frac{\beta}{2}\left(\frac{1}{\alpha}-1\right)} \cdot \|\nabla u\|_{\eta^2 A, 2}$$

$\square$

Proof of the main theorem:

(i) By Corollary, we can bound (2):

$$\sum_i \|u-u_i\|^2_{A_i \varphi_i 2\alpha} \leq \sum_i \varphi_i^{\frac{1}{\alpha}} r_i^{2+\beta\left(\frac{1}{\alpha}-1\right)} \cdot \|\nabla u\|^2_{\eta^2 A, 2}$$

which gives

$$\boxed{(2) \leq C \|\nabla u\|_2^2}$$

if $\sup \varphi_i^{\frac{1}{\alpha}} r_i^{2+\beta\left(\frac{1}{\alpha}-1\right)} \leq C$ ..... (*)

$_{i \in \mathbb{N}^*}$

As for 1, we bound it using Lemma 1 applied to the graph:

$$A_0 - A_1 - A_2 - A_3 - \cdots$$

with weights: $\omega_i = \dfrac{\tilde{\omega}_i}{\tilde{\omega}}$, where

$$\tilde{\omega}_i = \left(\varphi_i \, |A_i|\right)^{\frac{1}{\alpha}} \qquad \tilde{\omega} = \sum \tilde{\omega}_i$$

if the series converges.

Hence, if we choose:

$$u := \sum \omega_i u_i = \int u \varphi \, d\mathrm{vol}$$

$$\Psi := \sum \frac{w_i \chi_{\eta A_i}}{|\eta A_i|}, \quad \int \Psi \, d\mathrm{vol} = 1$$

, by Lemma 1, we get:

$$\underbrace{\sum \tilde{w}_i \overset{1}{|u_i - \mu|^2} \leq \frac{C}{\tilde{w}} \sum (\overline{w}_k + \overline{w}_{k+1}) \underbrace{|u_k - u_{k+1}|^2}}$$

$$\overline{w}_k := \sum_{i \leq k \leq j} (i-j) \tilde{w}_i \tilde{w}_j$$

$$\underbrace{|u_k - u_{k+1}|^2} = \left| \frac{1}{|\eta A_k|} \int_{\eta A_k} u - \frac{1}{|\eta A_{k+1}|} \int_{\eta A_{k+1}} u \right|^2$$

$$\frac{1}{\quad} \left| \int \frac{(u(x) - u(y))}{\quad} \right|^2$$

$$= \frac{\cdots}{(|\eta A_k||\eta A_{k+1}|)^2} \int_{\eta A_k \times \eta A_{k+1}} \cdots$$

Cauchy-Schwartz

$$\leq \frac{1}{|\eta A_k||\eta A_{k+1}|} \int_{\eta A_k \times \eta A_{k+1}} |u(x)-u(y)|^2$$

$$\leq \frac{2}{|\eta A_k||\eta A_{k+1}|} \left(|\eta A_{k+1}| \int_{\eta A_k} |u(x)-\lambda|^2 + |\eta A_k| \int_{\eta A_{k+1}} |u(y)-\lambda|^2 \right)$$

$$\leq \frac{2|\eta A_k \cup \eta A_{k+1}|}{|\eta A_k||\eta A_{k+1}|} \int_{\eta A_k \cup \eta A_{k+1}} |u(x)-\lambda|^2$$

$\eta A_k \cup \eta A_{k+1}$

$$\leq C r^{-\beta} \int_{\eta A_k \cup \eta A_{k+1}} |u(x) - \gamma|^2$$

Choosing γ to be the average over each region, using Lemma 2 and summing over k, we find that:

$$\boxed{① \leq C \|\nabla u\|^2} \quad \cdots (**)$$

if $C_0 \frac{1}{\,} < \tilde{w} < C$

$$\bullet\ \sup_k \left(\overline{\omega}_k + \overline{\omega}_{k+1}\right) r_k^{2-\beta} \leq C$$

Therefore by $*$ and $**$:

$$\|u-\mu\|_{M,\varphi,2\alpha} \leq C \|\nabla u\|_{M,2}$$

if :

$$\bullet\ \sup_{i\in\mathbb{N}_0} \varphi_i^{\frac{1}{\alpha}}\ r_i^{2+\beta\left(\frac{1}{\alpha}-1\right)} \leq C \quad \cdots 1$$

$$\bullet\ \frac{1}{C} \leq \left(\sum_j (\varphi_i |A_i|)\right)^{\frac{1}{\alpha}} \leq C \quad \cdots 2$$

$$\bullet\ \sup \left(\overline{\omega}_k + \overline{\omega}_{k+1}\right) r_k^{2-\beta} \leq C \quad \cdots 3$$

• $\sup_{k} \left(w_k \cdots w_{k+1} \right) \cdots \lesssim$

where $\overline{w}_k = \sum_{i \leq k \leq j} (j-i)\, \tilde{w}_i \tilde{w}_j$

$w_i = \left(\varphi_i |A_i| \right)^{\frac{1}{\alpha}}$

(1) checks if $\varphi \lesssim C(1+r)^{\alpha(\beta-2)-\beta}$

(2) checks if $\varphi|_{A_0} \gtrsim C^{-1}$ and

$\varphi \lesssim C(1+r)^{-\beta-\varepsilon}$ for some $\varepsilon > 0$

(3) checks if we have

$\varphi \lesssim C(1+r)^{\alpha(\min\{\beta-2,0\}-\varepsilon)-\beta}$