# Analysis for the complex Monge-Ampère equation

Let $(M,\omega_0)$ be a complete non-compact Kähler manifold satisfying $SOB(\beta)$, $\beta>0$
& with $C^{3,\alpha}$ quasi-atlas

## Tian-Yau-Hein package

Let $f\in C^{2,\alpha}(M)$ s.t. $|f|\leq C r^{-\mu}$ on $\{r>1\}$ for some $\mu>2$

If $\begin{cases}\beta>2 \\ \beta\leq 2 \ \& \ \int(e^f-1)\omega_0^n=0\end{cases}$, then $\exists\, \alpha\in(0,\bar\alpha]$, $u\in C^{4,\bar\alpha}(M)$ s.t. $(\omega_0+dd^c u)^n=e^f\omega_0^n$

Moreover for $\beta\leq 2$, $\int|\nabla u|^2\omega_0^n<+\infty$

Independent of the value of $\beta$, if $f\in C^{k,\alpha}_{loc}$ $\underset{k\geq 3}{\Rightarrow}$ $u\in C^{k+2,\alpha}_{loc}$

**Defn:** Let $(M,\omega_0)$ be a complete Kähler mfd.
A $C^{k,\alpha}$ quasi-atlas for $(M,\omega_0)$ is a collection $\{\Phi_x,\ x\in A\}$, $A\subset M$ of loc. holo. diffeo
$\Phi_x: \underset{B(0,1)\subset\mathbb{C}^n}{\overset{\shortparallel}{B}}\to M$, $\Phi_x(0)=x$ & $\exists\, C\geq 1$ with $\begin{cases}\mathrm{inj}(\Phi_x^*g_0)\geq 1/C \\ \frac{1}{C}g_{euc}\leq\Phi_x^*g_0\leq C g_{euc} \quad \forall x\in A \\ \|\Phi_x^* g_0\|_{C^{k,\alpha}(B,g_{euc})}\leq C\end{cases}$

& $\forall y\in M$, $\exists\, x\in A$ s.t. $y\in\Phi_x(B)$, $\mathrm{dist}_{g_0}(y,\partial\Phi_x(B))\geq 1/C$

↳ Given $C^{k,\alpha}$ quasi-atlas, define $\|u\|_{C^{l,r}(M)}:=\sup_{x\in A}\{\|u\circ\Phi_x\|_{C^{l,r}(B)}\}$

**Schauder:** for a 2nd order (unif) elliptic operator $L$ with coeff $\in C^{l,r}$

$$\|u\|_{C^{l+2,r}(M)}\leq C\left(\|Lu\|_{C^{l,r}(M)}+\|u\|_{L^\infty}\right)\quad\forall\ u\in C^\infty_{loc}(M)$$

( if $l\leq k-1$, $r\in(0,\alpha]$ )
↖ s.t. $C^{l,r}$-norm on $B$ is compariable w/ $C^{l,r}$-norm on $B$ induced by $\Phi_x^*g_0$

**Lemma (Tian-Yau)** If $|Rm|\leq C$, then $\exists\, C^{1,\alpha}$ quasi-atlas $\forall\,\alpha$
If moreover $\sum_{i=1}^k|\nabla^i Scal|\leq C$, $\exists\, C^{k+1,\alpha}$ quasi-atlas.
↗
we discuss its pf after pf of TYH

## § Existence I : $\varepsilon$-perturbed equation $(\omega_0+dd^c u_\varepsilon)^n=e^{f+\varepsilon u}\omega_0^n$ ... $(MA_\varepsilon)$

Continuity method by Cheng-Yau

$(M,\omega_0)$ complete Kähler manifold with $C^{2,\alpha}$ quasi-atlas, $f\in C^{2,\alpha}$

Goal : Show $\exists\, u_\varepsilon\in C^{4,\bar\alpha}$, for some $\bar\alpha\in(0,\alpha]$ to $(MA_\varepsilon)$
↳ depend on $M$, $\omega_0$, $\alpha$, $f$, but not on $\varepsilon$.

Consider $(\omega_0+dd^c u_{\varepsilon,t})^n=e^{tf+\varepsilon u_{\varepsilon,t}}\omega_0^n$ ... $(MA_{\varepsilon,t})$

For $r\in(0,\alpha]$, $J_r=\{t\in[0,1]\mid (MA_{\varepsilon,t})$ admits a sol'n $\in C^{4,r}\}$

$0\in J_r$, want to prove openness & closedness

**Lemma (Yau's maximum principle)**

Let $(M,g)$ be a complete mfd with sectional curvature bounded below, $x_0\in M$.
If $u\in C^2_{loc}$ with $|u|+|\nabla u|+|\nabla^2 u|\leq C$ does not attain its supremum, then $\exists\,\{x_k\}\subset M$,
with $\mathrm{dist}(x_0,x_k)\to\infty$ & s.t. $u(x_k)\to\sup u$, $|\nabla u|(x_k)\to 0$, $\limsup_k \max\mathrm{spec}(\nabla^2 u)(x_k)\leq 0$

Claim 1 : $\forall\, r \in (0,\alpha]$, $t \in J_r$, $\exists$ bdd linear operator $G : C^{2,r} \to C^{4,r}$ with $(\Delta_\omega - \varepsilon)\circ G = \mathrm{id}$

where $\omega = \omega_0 + dd^c u_{\varepsilon,t}$

↳ this gives the openness (linearized operator is an isomorphism ⇝ implicit fcn thm is good)

pf. Since $\varepsilon > 0$, $\forall$ bdd domain w/ sufficiently smooth bdry $\Omega \subset M$

$\exists\, u = u_\Omega \in C^{4,r}(\Omega)$ solves $\begin{cases} (\Delta - \varepsilon) u = f|_\Omega \\ u = 0 \text{ on } \partial\Omega \end{cases}$

Schauder ⇝ there are local estimates (w.r.t. charts of quasi-atlas) for $\|u\|_{C^{4,r}}$ in terms of $\|f\|_{C^{2,r}}$ & $\|u\|_{L^\infty}$

Max prin : at max $x_{max} \in \Omega$, $f(x_{max}) = (\Delta - \varepsilon) u|_{x_{max}} \le -\varepsilon\, u(x_{max}) \Rightarrow u(x_{max}) \le -\frac{1}{\varepsilon} f(x_{max})$

similarly, $u(x_{min}) \ge -\frac{1}{\varepsilon} f(x_{min})$

$\Rightarrow \|u\|_{L^\infty} \le \frac{1}{\varepsilon}\|f\|_{L^\infty}$

Arzela-Ascoli : $\{u_{\Omega_k}\} \to u$ s.t. $(\Delta_\omega - \varepsilon) u = f$ on $M$

uniqueness : for $(\Delta - \varepsilon) u = 0$ on $M$

if $x_{max} \in M \Rightarrow u(x_{max}) \le 0$
$x_{min} \in M \Rightarrow u(x_{min}) \ge 0$

| if sup not in $M$, use Yau's max prin
inf

#

Claim 2 : $\exists\, \bar\alpha \in (0,\alpha]$ s.t. if $u = u_{\varepsilon,t} \in C^4$ solves $(MA_{\varepsilon,t})$ for $t \in [0,1]$, $\varepsilon \in (0,1]$,

then $\|u\|_{C^{2,\bar\alpha}(M)} \le C(M, \omega_0, \|f\|_{C^{2,\alpha}(M)}, \varepsilon)$

↳ this gives the closedness.

pf : By max prin (or Yau's max prin), $\|u\|_{L^\infty} \le \frac{1}{\varepsilon}\|f\|_{L^\infty}$

$C^2$ : Yau $0 \le \mathrm{tr}_{\omega_0}\omega = n + \Delta_{\omega_0} u \le C\cdot \exp\left(C(u - \inf u)\right)$, $C = C(M, \omega_0, \|f\|_{C^2(M)})$ if $\varepsilon \le 1$

↳ $\frac{1}{C}\omega_0 \le \omega \le C\omega_0$

Bounded $\Delta$ + Linear theory $\Rightarrow \|u\|_{C^{1,r}(M)} \le C(r)$ $\forall\, r \in (0,1)$

Schauder ⇝ $u \in C^{2,\alpha}$ uniformly #

## § Existence 2 : the limit $\varepsilon \to 0$

Rmk : only need to make $L^\infty$-estimate unif in $\varepsilon$

Step 1 : high integrability of the sol'ns ($L^p$ might still blow up as $\varepsilon \to 0$)

Step 2 : Moser iteration

Recall : $\omega^n - \omega_0^n = dd^c u \wedge \underbrace{\textstyle\sum_{k=0}^{n-1} (\omega^k \wedge \omega_0^{n-1-k})}_{= T}$ ... ⊛

For $u = u_\varepsilon \in C^2(M)$ solving $(MA_\varepsilon)$, $\zeta \in C_c^\infty(M)$

Step 1: ⊛ $\times\ \zeta u|u|^{p-2}$, $p > 1$, $\quad \zeta u|u|^{p-2}(e^{f+\varepsilon u} - 1)\,\omega_0^n = \zeta u|u|^{p-2} dd^c u \wedge T$

$$\int \zeta u|u|^{p-2} dd^c u \wedge T = -\int u|u|^{p-2} d\zeta \wedge d^c u \wedge T - (p-1)\int \zeta |u|^{p-2} du \wedge d^c u \wedge T$$

$$\begin{aligned} \int \zeta \left|\nabla |u|^{\frac{p}{2}}\right|^2_{\omega_0} \omega_0^n &= n\frac{p^2}{4}\int \zeta |u|^{p-2} du \wedge d^c u \wedge \omega_0^{n-1} \\ &\le n\frac{p^2}{4}\int \zeta |u|^{p-2} du \wedge d^c u \wedge T \\ &= \frac{-np^2}{4(p-1)}\Big\{\int \zeta u|u|^{p-2} \underbrace{dd^c u \wedge T}_{= (e^{f+\varepsilon u} - 1)\,\omega_0^n} + \int u|u|^{p-2} d\zeta \wedge d^c u \wedge T\Big\} \end{aligned}$$

$$\Rightarrow \int \xi \,|\nabla |u|^{p/2}|^2_{\omega_0}\, \omega_0^n + \frac{np^2}{4(p-1)} \int \xi\, u|u|^{p-2}(e^{\varepsilon u}-1)\, e^f \omega_0^n$$

$$\leq -\frac{np^2}{4(p-1)} \left\{ \int \xi\, u|u|^{p-2}(e^f-1)\,\omega_0^n + \int u|u|^{p-2} d\xi \wedge d^c u \wedge T \right\}$$

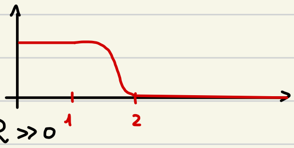
Note: $\forall C<\infty,\ \exists\, \delta=\delta(C)>0$ s.t. if $|x|\leq C$, $x(e^x-1) \geq \delta x^2$

$\varepsilon \|u\|_{L^\infty} \leq \|f\|_{L^\infty} \Rightarrow u(e^{\varepsilon u}-1) \geq \delta \varepsilon u^2$

(1) High integrability

Schoen-Yau : if $\mathrm{Ric}(\omega_0) \geq -C \Rightarrow \exists$ smooth $\rho : M \to \mathbb{R}$ s.t. $\rho \sim 1+\mathrm{dist}(x_0,\cdot)$ for any fixed $x_0$ & $|\nabla\rho| + \rho|\Delta\rho| \leq C$

Fix a cutoff $\chi : \mathbb{R}_+ \to \mathbb{R}_+$



$\xi_k = \chi(\rho/R)\cdot\rho^k,\ k\in\mathbb{R},\ R\gg 0$

$$\hookrightarrow \int \xi_k\, |\nabla |u|^{p/2}|^2_{\omega_0}\,\omega_0^n + \frac{np^2}{4(p-1)} \underbrace{\int \xi_k\, u|u|^{p-2}(e^{\varepsilon u}-1)\, e^f \omega_0^n}_{\geq\ \delta\varepsilon \int \xi_k |u|^p e^f \omega_0^n} \quad \cdots (**)$$

$$\leq -\frac{np^2}{4(p-1)} \left\{ \int \xi_k\, u|u|^{p-2}(e^f-1)\,\omega_0^n + \underbrace{\int u|u|^{p-2} d\xi_k \wedge d^c u \wedge T}_{\lesssim \int |u|^{p-1}\rho^{k-1}|du|\,\omega_0^n} \right\}$$

$d\xi_k = \chi'(\rho/R)\,\rho^k \frac{d\rho}{R} + \chi(\rho/R)\, k\, \rho^{k-1} d\rho$

$|d\xi_k| \lesssim (|k|+1)\,\rho^{k-1}|d\rho|$

$\lesssim a \int \rho^k |u|^{p-2}|du|^2 \omega_0^n + \frac{1}{a}\int \rho^{k-2}|u|^p \omega_0^n$

(depends on $\varepsilon$)

$T \approx \omega_0^{n-1}$

Assume $\|u\|_{C^{2,\alpha}} < +\infty$, $\|f\|_{L^\infty} < +\infty$ as $R\to+\infty$

$$\int \rho^k |\nabla|u|^{p/2}|^2_{\omega_0}\,\omega_0^n + \frac{np^2}{4(p-1)}\,\delta\cdot\varepsilon \int \rho^k |u|^p \omega_0^n \leq \frac{np^2}{4(p-1)} \left\{ \int \rho^k |u|^{p-1} |e^f-1|\,\omega_0^n + \frac{C(k)}{p}\int \rho^{k-2}|u|^p\omega_0^n \right\}$$

$(M,\omega_0)$ has poly vol growth $|B_{\omega_0}(x_0,s)| \leq C s^\beta$

Temporary assumption : $f \in C_c^\infty(M)$

For $k_0\in\mathbb{Z}$ sufficiently negative s.t. $\int \rho^{k_0-2}|u|^p < +\infty$ & $0<a\ll 1$ and put $a\cdot\int \rho^k|u|^{p-2}|du|^2\omega_0^n$ on RHS.

$$\hookrightarrow \int \rho^k |u|^{p-2}|\nabla u|^2 + \varepsilon \int \rho^k |u|^p \lesssim \int \rho^k |u|^p |e^f-1| + \int \rho^{k-2}|u|^p$$

Starting from $k_0$, can iterate

$$\Rightarrow \int \rho^k |u|^{p-2}|\nabla u|^2 + \int \rho^k |u|^p < +\infty \quad \forall k\in\mathbb{N}_0,\ p>1$$

## (2) Uniform $L^\infty$-estimate

Take $\zeta = \zeta_0 = \chi(\rho/R)$ $\rightsquigarrow$ (A) with $k=0$, $R\to\infty$ & drop the $\varepsilon$ term on the LHS:

$$\int |\nabla |u|^{p/2}|^2_{\omega_0}\,\omega_0^n \le -\frac{np^2}{4(p-1)}\int u|u|^{p-2}(e^f-1)\,\omega_0^n$$

$\boxed{\beta>2}$: SOB($\beta$), $\beta>2$ $\rightsquigarrow$

$$\underbrace{\left(\int \rho^{\alpha(\beta-2)-\beta}\,\big||u|^{p/2}\big|^{2\alpha}\omega_0^n\right)^{1/\alpha}}_{=\|u\|^p_{\alpha p,\alpha}} \le C\int|\nabla|u|^{p/2}|^2_{\omega_0}\,\omega_0^n \le \frac{Cnp^2}{(p-1)}\int |u|^{p-1}|e^f-1|\,\omega_0^n \quad \cdots \text{(I)}$$

$\|v\|_{p,\alpha} := \left(\int \rho^{\alpha(\beta-2)-\beta}|v|^p\right)^{1/p}$

note that $|e^f-1|\le C\cdot|f|$
↑ depend on $\sup|f|$

**step I** (I) $\Rightarrow$

$$\|u\|^p_{\alpha p,\alpha} \le \frac{Cnp^2}{(p-1)}\int|u|^{p-1}\rho^{-\mu}\,\omega_0^n \le \frac{Cnp^2}{(p-1)}\underbrace{\left(\int |u|^{\overset{m(p-1)}{\overset{\shortparallel}{\alpha p}}}\rho^{m\zeta}\right)^{1/m}}_{=\|u\|^{p-1}_{\alpha p,\alpha}}\underbrace{\left(\int\rho^{-m^*\zeta}\rho^{-m^*\mu}\right)^{1/m^*}}_{=A}$$

need $m(p-1)=\alpha p$ $\Rightarrow$ $m=\frac{\alpha p}{p-1}$, $m^*=\frac{\alpha p}{\alpha p-p+1}$

$m\zeta = \alpha(\beta-2)-\beta$ $\Rightarrow$ $\zeta=\frac{p-1}{\alpha p}(\alpha(\beta-2)-\beta)$

If $r>\beta$, $\int\rho^{-r}<C$. So we want $m^*(\zeta+\overset{s}{\overset{\shortparallel}{\mu}})\ge\beta+\varepsilon$

$\Leftrightarrow$ $p\alpha(\zeta+\mu)\ge(\beta+\varepsilon)(p(\alpha-1)+1)$

$\Leftrightarrow$ $p\{\alpha(\beta-2)-\beta+\alpha\mu\}-\alpha(\beta-2)+\cancel{\beta} \ge p(\beta+\varepsilon)(\alpha-1)+\cancel{\beta}+\varepsilon$

$\Leftrightarrow$ $\underbrace{p\{\alpha\mu+\alpha(\beta-2)-\beta-(\beta+\varepsilon)(\alpha-1)\}}_{=p\{\alpha(\mu-2)-\varepsilon(\alpha-1)\}}\ge\alpha(\beta-2)+\varepsilon$

$\Leftrightarrow$ $p\ge\frac{\alpha(\beta-2)+\varepsilon}{\alpha(\mu-2)-\varepsilon(\alpha-1)}$ if $\varepsilon$ sufficiently small s.t. $(\mu-2)-\varepsilon(\alpha-1)>0$

↳ if $p>\frac{(\beta-2)+\varepsilon}{(\mu-2)-\varepsilon(\alpha-1)}$, $A\le\left(\int\rho^{-(\beta+\varepsilon)}\right)^{1/m^*}$

$$\int\rho^{-(\beta+\varepsilon)}\sim\sum_{i=0}^\infty\int_{A(\kappa^i r_0,\,\kappa^{i+1}r_0)}\rho^{-(\beta+\varepsilon)}\lesssim\sum_{i=0}^\infty(\kappa^{i+1}r_0)^\beta\cdot(\kappa^i r_0)^{-(\beta+\varepsilon)} = \kappa^\beta r_0^{-\varepsilon}\sum_{i=0}^\infty\kappa^{-i\varepsilon}$$

$$\sim r_0^{-\varepsilon} = \exp(-\varepsilon\log r_0)\sim\frac{1}{\varepsilon}$$

$\frac{1}{m^*}=\frac{p(\alpha-1)+1}{\alpha p}$ ↘ in $p$ $\Rightarrow$ $\left(\int\rho^{-(\beta+\varepsilon)}\right)^{1/m^*}\lesssim\frac1\varepsilon$ if $p>1$, $p>\frac{\beta-2+\varepsilon}{\mu-2-\varepsilon(\alpha-1)}$

All in all $\|u\|_{\alpha p,\alpha}\le\frac{Cnp^2}{\varepsilon(p-1)}$ ... (1)

**step II** (I) $\Rightarrow$

$$\|u\|^p_{\alpha p,\alpha}\le\frac{Cp^2}{(p-1)}\int|u|^{p-1}\rho^{-\mu}\le\frac{Cp^2}{p-1}\underbrace{\left(\int|u|^{\overset{m(p-1)}{\overset{\shortparallel}{p}}}\rho^{m\eta}\right)^{1/m}}_{=\|u\|^{p-1}_{p,\alpha}}\left(\int\rho^{-m^*\eta}\rho^{-m^*\mu}\right)^{1/m^*}$$

need $m(p-1)=p$ $\Rightarrow$ $m=\frac{p}{p-1}$, $m^*=p$

$m\eta=\alpha(\beta-2)-\beta$ $\Rightarrow$ $\eta=\frac{p-1}{p}(\alpha(\beta-2)-\beta)$

Want $m^*(\eta+\mu)\ge\beta+\varepsilon$ $\Leftrightarrow$ $(p-1)(\alpha(\beta-2)-\beta)+p\mu\ge\beta+\varepsilon$

$\Leftrightarrow$ $p\{\mu+\alpha(\beta-2)-\beta\}-\alpha(\beta-2)+\cancel\beta\ge\cancel\beta+\varepsilon$

$\Leftrightarrow$ $p\ge\frac{\alpha(\beta-2)+\varepsilon}{\mu+\alpha(\beta-2)-\beta}$ if $\mu+\alpha(\beta-2)-\beta>0$ true if $\alpha$ close to 1

Then $\|u\|_{\alpha p,\alpha}\le\left(\frac{Cp^2}{\varepsilon(p-1)}\right)^{1/p}\|u\|^{1-1/p}_{p,\alpha}$ if $p>\frac{\alpha(\beta-2)+\varepsilon}{\mu+\alpha(\beta-2)-\beta}$ & $\underset{[1,\frac{n}{n-1}]}{\underset{\uparrow}{\alpha}}$ s.t. $\mu+\alpha(\beta-2)-\beta>0$

Step I + II + Iteration $\Rightarrow$ $\|u\|_{L^\infty}\le C$ : indep of $\varepsilon$

Decay estimate for $\beta > 2$: Suppose that $|d\rho| + \rho|dd^c\rho| \leq C$.
If $\mu \in (0,\beta) \Rightarrow \forall\,\delta > 0,\ |u| \leq C(\delta)\,\rho^{2-\mu+\delta}$

$$\zeta = \chi(\rho/R)\cdot\rho^{l}$$

$$\bigstar \times \zeta u|\zeta u|^{p-2}\zeta \Rightarrow \int \zeta u|\zeta u|^{p-2}\zeta\,(e^{f+\varepsilon u}-1)\,\omega_0^n = \int \zeta u|\zeta u|^{p-2}\zeta\, dd^c u\wedge T$$

$$\int |\nabla|\zeta u|^{p/2}|^2_{\omega_0}\,\omega_0^n = n\int d|\zeta u|^{p/2}\wedge d^c|\zeta u|^{p/2}\wedge\omega_0^{n-1}$$
$$= n\int |u|^p\cdot\left(\tfrac{p}{2}\zeta|\zeta|^{\frac{p}{2}-2}\right)^2 d\zeta\wedge d^c\zeta\wedge\omega_0^{n-1} + \underline{n\int \zeta^p\left(\tfrac{p}{2}u|u|^{\frac{p}{2}-2}\right)^2 du\wedge d^cu\wedge\omega_0^{n-1}}$$
$$+2n\int \tfrac{p^2}{4}\,\zeta u|\zeta u|^{p-2}\,d\zeta\wedge d^cu\wedge\omega_0^{n-1}$$

$$\lesssim \frac{p^2}{4}\int \zeta^p|u|^{p-2}\,du\wedge d^cu\wedge T$$
$$= \frac{p^2}{4(p-1)}\int \zeta^p\, d|u|^{p-1}\wedge d^cu\wedge T$$
$$= -\frac{p^2}{4(p-1)}\left\{\begin{aligned}&\int |\zeta u|^{p-1}\zeta\, dd^cu\wedge T\\ &+p\int |\zeta u|^{p-1}\,d\zeta\wedge d^cu\wedge T\end{aligned}\right\}$$

$$\leq -\frac{np^2}{4(p-1)}\int |\zeta u|^{p-1}\zeta\,(e^{f+\varepsilon u}-1)\,\omega_0^n + \frac{np^2}{4}\int|\zeta u|^{p-2}|u|^2\,d\zeta\wedge d^c\zeta\wedge\omega_0^{n-1}$$
$$+\frac{np^2}{2}\underbrace{\int \zeta u|\zeta u|^{p-2}\,d\zeta\wedge d^cu\wedge\omega_0^{n-1}}_{\text{similar}} - \underline{\frac{np^2}{4(p-1)}\int |\zeta u|^{p-1}\,d\zeta\wedge d^cu\wedge T}$$

$$= -\frac{np^2}{4(p-1)}\left\{\begin{aligned}&\int (p-1)\,|\zeta u|^{p-2}|u|^2\,d\zeta\wedge d^c\zeta\wedge T\\ &+\int |\zeta u|^{p-1}u\,dd^c\zeta\wedge T\end{aligned}\right\}$$

Note: ① $\varepsilon\|u\|_{L^\infty} \leq \|f\|_{L^\infty} \Rightarrow u(e^{\varepsilon u}-1) \geq \delta\cdot\varepsilon u^2 \geq 0$

② $\|u\|_{L^\infty} \leq C$ indep of $\varepsilon$ $\Rightarrow$ Yau's $C^2$: $\omega \approx \omega_0$; thus, $T \approx \omega_0^{n-1}$

③ $d\zeta = \chi'(\rho/R)\,\rho^l\,\frac{d\rho}{R} + \chi(\rho/R)\,l\,\rho^{l-1}d\rho \Rightarrow d\zeta\wedge d^c\zeta \lesssim (|l|+1)^2\rho^{2l-2}\omega_0$

$dd^c\zeta = \chi''(\rho/R)\,\rho^l\,\frac{d\rho\wedge d^c\rho}{R^2} + \cdots \Rightarrow |dd^c\zeta|_{\omega_0} \lesssim (|l|+1)^2\rho^{2l-2}$

$$\hookrightarrow \int |\nabla|\zeta u|^{p/2}|^2_{\omega_0}\,\omega_0^n + \frac{np^2}{4(p-1)}\int \zeta^2|\zeta u|^{p-2}u\,(e^{\varepsilon u}-1)\,e^f\omega_0^n$$
$$\leq \frac{-np^2}{4(p-1)}\int \zeta u|\zeta u|^{p-2}\zeta\,(e^f-1)\,\omega_0^n + Cnp^2(|l|+1)^2\int|\rho^l u|^p\rho^{-2}\,\omega_0^n$$

$$\hookrightarrow \int |\nabla|\zeta u|^{p/2}|^2_{\omega_0}\,\omega_0^n \leq \frac{np^2}{4(p-1)}\left\{\int |\zeta u|^{p-1}\underbrace{|e^f-1|}_{\lesssim C\rho^{-\mu}}\,\omega_0^n + Cp(|l|+1)^2\int\rho^{-2}|\rho^l u|^p\,\omega_0^n\right\}$$

VS $\leftarrow$ SOB($\beta$), $\beta>2$

$$\left(\int \rho^{\alpha(\beta-2)-\beta}|\zeta u|^{p\alpha}\,\omega_0^n\right)^{1/\alpha}$$

$$\eta := \rho^l \Rightarrow \left(\int \rho^{\alpha(\beta-2)-\beta}|\eta u|^{p\alpha}\right)^{1/\alpha} \lesssim \frac{p^3}{p-1}\left(\int|\eta u|^{p-1}\rho^{-\mu} + (|l|+1)^2\int\rho^{-2}|\eta u|^p\right)$$

Now $\eta_k = \rho^{l_k}$, $l_k$: TBD, $p_k = \alpha^k p_0$, $k\in\mathbb{N}$

$$\text{Then } \Bigg(\underbrace{\int \rho^{\alpha(\beta-2)-\beta+l_k\alpha^{k+1}p_0}}_{=C}\,|u|^{\alpha^{k+1}p_0}\Bigg)^{1/\alpha} \lesssim \frac{p_k^3}{p_k-1}\left(\overbrace{\int \rho^{l_k\alpha^k p_0 - l_k - \mu}\,|u|^{\alpha^k p_0-1}}^{A} + (|l_k|+1)^2\underbrace{\int\rho^{l_k\alpha^k p_0-2}\,|u|^{\alpha^k p_0}}_{B}\right)$$

For $l_k \le \frac{\beta-2}{p_0}\left(1-\frac{1}{\alpha^k}\right) \Rightarrow l_k \alpha^k p_0 - 2 \le \alpha^k(\beta-2)-\beta$

$$\Rightarrow B \le \int \rho^{\alpha^k(\beta-2)-\beta}|u|^{\alpha^k p_0}$$

$l_k \ge \frac{\beta-2}{p_0}\left(1-\frac{1}{\alpha^k}\right) \Rightarrow l_k\alpha^{k+1}p_0 + \alpha(\beta-2)-\beta \ge \alpha^{k+1}(\beta-2)-\beta$

$$\Rightarrow C \ge \int \rho^{\alpha^{k+1}(\beta-2)-\beta}|u|^{\alpha^{k+1}p_0}$$

Take $l_k = \frac{\beta-2}{p_0}\left(1-\frac{1}{\alpha^k}\right)$

$$A = \int |\rho^{l_k}u|^{p_k-1}\cdot\rho^{-\mu} \le \left(\int|\rho^{l_k}u|^{p_k}\rho^{-2}\right)^{\frac{p_k-1}{p_k}}\left(\int \rho^{2(p_k-1)}\rho^{l_k p_k}\rho^{-p_k\mu}\right)^{1/p_k}$$

$$\le \max\{1, B\}\left(\int\rho^{p_k(2+l_k-\mu)-2}\right)^{1/p_k}$$

If $p_0(2-\mu)+\beta-2 = -\varepsilon<0 \Rightarrow p_k(2+l_k-\mu)-2 = \alpha^k p_0\left(2+\left(\frac{\beta-2}{p_0}\right)\left(1-\frac{1}{\alpha^k}\right)-\mu\right)-2 = -\alpha^k\varepsilon-\beta$

$$\Rightarrow \int\rho^{p_k(2+l_k-\mu)-2} = \int\rho^{-\beta-\alpha^k\varepsilon} \le C \quad \forall k\ge1$$

$$\Rightarrow \int\rho^{-\beta}\left|\rho^{\frac{\beta-2}{p_0}}u\right|^{\alpha^{k+1}p_0} \lesssim \left(\frac{p_k^3}{p_k-1}\left(\frac{\beta-2+p_0}{p_0}\right)^2\right)^{1/p_k}\max\left\{1, \int\rho^{-\beta}\left|\rho^{\frac{\beta-2}{p_0}}u\right|^{\alpha^k p_0}\right\}$$

Recall that $\|u\|_{\alpha p,\alpha} \le \frac{Cnp^2}{\varepsilon(p-1)}$ if $p>1$, $p>\frac{\beta-2+\varepsilon}{\mu-2-\varepsilon(\alpha-1)}$, $\mu-2-\varepsilon(\alpha-1)>0$

$$\|u\|_{\alpha p,\alpha} = \left(\int\rho^{\alpha(\beta-2)-\beta}|u|^{\alpha p}\right)^{1/\alpha p} = \left(\int\rho^{-\beta}\left|\rho^{\frac{\beta-2}{p}}u\right|^{\alpha p}\right)^{1/\alpha p}$$

take $p_0>1$ s.t. $p_0 > \frac{\beta-2+\varepsilon}{\mu-2-\varepsilon(\alpha-1)}$ & $p_0(2-\mu)+\beta-2 = -\delta<0$ $\quad \exists\varepsilon$ s.t.

↳ Moser iteration $\Rightarrow \left|\rho^{\frac{\beta-2}{p_0}}u\right| \le C$ & if $2<\mu<\beta$, we can always find such a $p_0$ #

Decay estimate 2: Same assumption on $\rho$, if $\mu\ge\beta$, $\begin{cases} \text{if } \frac{\beta}{\mu-2}<1 & |u| \le C_\delta\rho^{-(\beta-2)+\delta} \\ \text{if } \frac{\beta}{\mu-2}\ge1, \forall\delta>0, & |u|\le C_\delta\rho^{-\frac{(\beta-2)(\mu-2)}{\beta+\delta}} \end{cases}$

we don't have rapidly decay for $\mu\gg1$.

We only need to check $p_k(2-l_k-\mu)-2 \le -\beta-\delta$ for some $\delta>0$

$$p_k(2-l_k-\mu)-2 = \alpha^k p_0\left(2-\mu+\frac{\beta-2}{p_0}\left(1-\frac{1}{\alpha^k}\right)\right) = \alpha^k\left(p_0(2-\mu)+\beta-2\right)-\beta+2$$

$\Leftrightarrow \alpha^k\left(p_0(2-\mu)+\beta-2\right)\le -2-\delta \quad \forall k$

for $\varepsilon\ll1$, this is fine

$\Leftrightarrow p_0(2-\mu)+\beta-2 \le -2-\delta \Leftrightarrow p_0(2-\mu)+\beta\le-\delta \Leftrightarrow p_0\ge\frac{\beta+\delta}{\mu-2}$ $(\mu\ge\beta>2)$

Take $p_0>1$, $p_0>\frac{\beta-2+\varepsilon}{\mu-2-\varepsilon(\alpha-1)}$, $p_0(2-\mu)+\beta-2\le-2-\delta$

↳ $\left|\rho^{\frac{\beta-2}{p_0}}u\right|\le C \Rightarrow |u|\le C\rho^{-\frac{\beta-2}{p_0}}$ $\begin{cases} \text{if } \frac{\beta+\delta}{\mu-2}\le1, \text{ take arbitrary } p_0>1 \rightsquigarrow |u|\lesssim\rho^{-(\beta-2)+\varepsilon'} \\ \text{if } \frac{\beta+\delta}{\mu-2}>1, \text{ take } p_0=\frac{\beta+\delta}{\mu-2} \rightsquigarrow |u|\lesssim\rho^{-\frac{(\beta-2)(\mu-2)}{\beta+\delta}} \end{cases}$