$\boxed{\beta\in(0,2]}$ Claim 1: $\int (e^{f+\varepsilon u}-1)\,\omega_0^n = 0$

pf. $\zeta_R = \chi(\rho/R)$ $\times$ ⊛ $\Rightarrow \int \eta_R (e^{f+\varepsilon u}-1)\,\omega_0^n = n\int \eta_R\, dd^c u\wedge T = -n\int du\wedge d^c\eta_R\wedge T$

$|d\eta_R| \lesssim 1/R$ $\rightsquigarrow |LHS| \lesssim \frac{1}{R}\int |\nabla u|\,\omega_0^n$ ($\omega_0\approx\omega$, but dep on ε)

We want to show $\int |\nabla u|\,\omega_0^n < +\infty$

Recall: $\int \rho^k |u|^{p-2}|\nabla u|^2 + \int \rho^k |u|^p < +\infty$

$$\left.\begin{array}{ll} k=3 & \Rightarrow \int \rho^{-k} < +\infty \\ p=2 & \Rightarrow \int \rho^{-k}|\nabla u|^2 < +\infty \end{array}\right\} \Rightarrow \int |\nabla u| \leq \underbrace{\left(\int \rho^k |\nabla u|^2\right)^{1/2}\left(\int \rho^{-k}\right)^{1/2}}_{<+\infty}$$

#

Claim 2: $\exists\, C>0$ indep of ε s.t. $\left(\int \rho^{\alpha(\beta-2-\delta)-\beta}|u-u_{\psi_\delta}|^{2\alpha}\right)^{1/2\alpha} \lesssim C(\delta)$ for α close to 1

pf Put $v = u - u_{\psi_\delta}$

$u\times$ ⊛ $\Rightarrow \int u\,(e^{f+\varepsilon u}-1)\,\omega_0^n = n\int u\, dd^c u\wedge T = -n\int du\wedge d^c u\wedge T$

$\Rightarrow \int dv\wedge d^c v\wedge T = \int u(1-e^f)\,\omega_0^n + \int u\,(1-e^{\varepsilon u})\,e^f\omega_0^n$, recall: $u\,(e^{\varepsilon u}-1)\gtrsim \varepsilon u^2 \geq 0$

$\Rightarrow \int |\nabla v|^2\,\omega_0^n \leq \int |v|\,|e^f-1|\,\omega_0^n$ since $\int u(1-e^f) = \int (u-u_{\psi_\delta})(1-e^f)\,\omega_0^n$

$\Uparrow$

$\int (e^f-1)\,\omega_0^n = 0$

SOB(β), $\beta\in(0,2]$

$\hookrightarrow \left(\int \rho^{\alpha(\beta-2-\delta)-\beta}|v|^{2\alpha}\omega_0^n\right)^{1/2\alpha} \leq C(\delta)\int |\nabla v|^2\omega_0^n \leq C_\delta \int |v|\,|e^f-1|\,\omega_0^n$

$\leq C_\delta \int |v|\,\rho^{-\mu}\omega_0^n$

$\int |v|\rho^{-\mu} \leq \left(\int |v|^{2\alpha}\rho^{m\eta}\right)^{1/m}\left(\int \rho^{-m^*\eta}\rho^{-m^*\mu}\right)^{1/m^*}$

need $m = 2\alpha$, $m\eta = \alpha(\beta-2-\delta)-\beta$

$\Rightarrow m^* = \frac{2\alpha}{2\alpha-1}$, $\eta = \frac{\alpha(\beta-2-\delta)-\beta}{2\alpha}$

$\left(\int \rho^{\alpha(\beta-2-\delta)-\beta}|v|^{2\alpha}\right)^{1/2\alpha} \lesssim \left(\int \rho^{-m^*(\eta+\mu)}\right)^{1/m^*}$

If $m^*(\eta+\mu) \geq \beta+\delta'$ $\Rightarrow \left(\int \rho^{\alpha(\beta-2-\delta)-\beta}|v|^{2\alpha}\right)^{1/2\alpha} \leq C_{\delta,\delta'}$

$\Updownarrow$

#

$\alpha(\beta-2-\delta)-\beta+2\alpha\mu \geq (2\alpha-1)(\beta+\delta)$

$\Leftrightarrow \alpha(2(\beta+\delta')-\beta+2+\delta-2\mu) \leq \delta'$, true for α close to 1

$\boxed{\text{Lemma 1}}$ $\exists$ constant $C>0$ indep of ε ($\approx$ supp f) s.t.

(1) If $u_{\psi_\delta}\geq 0$, $-C\leq u\leq u_{\psi_\delta}+C$

(2) If $u_{\psi_\delta}\leq 0$, $-C+u_{\psi_\delta}\leq u\leq C$

pf. We show (1) via Moser iteration.

Set $v_+ = (u-u_{\psi_\delta})_+$.

When $v_+>0$, $u>u_{\psi_\delta}\geq 0$

$\Rightarrow v_+(e^{\varepsilon u}-1)\geq 0$ on M

$\hookrightarrow \int |\nabla v_+^{p/2}|^2_{\omega_0}\,\omega_0^n \leq \frac{np^2}{4(p-1)}\int |v_+|^{p-1}|e^f-1|\,\omega_0^n$

SOB(β), $\beta \leq 2$

$$\hookrightarrow \left(\int \rho^{\alpha(\beta-2-\delta)-\beta} \left| v_+^{\frac{p}{2}} - (v_+^{\frac{p}{2}})_{\psi_\delta} \right|^{2\alpha}\right)^{1/\alpha} \leq C_\delta \int |\nabla v_+^{\frac{p}{2}}|^2_{\omega_0} \omega_0^n$$

$$\leq C_\delta \frac{np^2}{4(p-1)} \int |v_+|^{\frac{p}{2}} |e^f - 1| \omega_0^n$$

$$\leq C_\delta \frac{np^2}{4(p-1)} \int |v_+|^{\frac{p}{2}} \rho^{-\mu} \omega_0^n$$

Note: Set $\|g\|_{p,\alpha,\delta} := \left(\int |g|^p \rho^{\alpha(\beta-2-\delta)-\beta}\right)^{1/p}$

$$\hookrightarrow \|(|v|^{\frac{p}{2}})_{\psi_\delta}\|_{2\alpha,\alpha,\delta} = \left(\int_M \left\{ \int |v|^{\frac{p}{2}} \cdot \psi_\delta \times \left(\int \psi_\delta\right)^{-1} \right\}^{2\alpha} \cdot \rho^{\alpha(\beta-2-\delta)-\beta}\right)^{1/2\alpha}$$

$$= \underline{\left(\int \psi_\delta\right)^{-1} \left(\int \rho^{\alpha(\beta-2-\delta)-\beta}\right)^{1/2\alpha}} \; \underline{\int |v|^{\frac{p}{2}} \cdot \psi_\delta}$$

$$\leq \underline{\left(\int \frac{\psi_\delta^2}{\rho^{\alpha(\beta-2-\delta)-\beta}}\right)^{1/2}} \|v\|^{\frac{p}{2}}_{p,\alpha,\delta}$$

product of these 3 $\lesssim C/\eta\alpha(2-\beta-\delta)$ if $1 \leq \alpha < 2 - \frac{\eta}{2-\beta+\delta}$

$$\Rightarrow \|v_+\|_{\alpha p,\alpha,\delta} \leq \left(\frac{C_{\alpha,\delta,\eta}\, p^2}{p-1}\right)^{1/p} \|v_+\|^{1-1/p}_{p,\alpha,\delta} + C^{1/p}_{\alpha,\delta,\eta} \|v_+\|_{p,\alpha,\delta}$$

w/ $\alpha(2-\beta+\delta) \leq \mu - \beta$

Then we can iterate from $p_0 = 2\alpha$ (Claim 2) $\Rightarrow |v_+| \leq C_{\text{unif}}$

$\Rightarrow u \leq u_{\psi_\delta} + C$

To get "$-C \leq u$", we replace v_+ in the above argument by $u_- = \min\{u, 0\}$

$\Rightarrow u_- \cdot (e^{\varepsilon u} - 1) \geq 0$ ⋯ (a)

Similarly by SOB(β), $\beta \leq 2$, we have

$$\left(\int \rho^{\alpha(\beta-2-\delta)-\beta} \left| |u_-|^{\frac{p}{2}} - (|u_-|^{\frac{p}{2}})_{\psi_\delta} \right|^{2\alpha}\right)^{1/2\alpha} \leq C_\delta \int |\nabla |u_-|^{\frac{p}{2}}|^2_{\omega_0} \omega_0^n$$

$$\leq C_\delta \frac{np^2}{4(p-1)} \int |u_-|^{p-1} |e^f - 1| \omega_0^n \quad \cdots \text{(b)}$$

With (a) + (b), argue as before

$$\Rightarrow \|u_-\|_{\alpha p,\alpha,\delta} \leq \left(\frac{C_{\alpha,\delta,\eta}\, p^2}{p-1}\right)^{1/p} \|u_-\|^{1-1/p}_{p,\alpha,\delta} + C^{1/p}_{\alpha,\delta,\eta} \|u_-\|_{p,\alpha,\delta}$$

with $\alpha(2-\beta+\delta) \leq \mu - \beta$

Since $u_{\psi_\delta} \geq 0 \Rightarrow (u-u_{\psi_\delta})_- \leq u_-$

Claim 2: $\left(\int \rho^{\alpha(\beta-2-\delta)-\beta} |u-u_{\psi_\delta}|^{2\alpha}\right)^{1/2\alpha} \leq C_\delta$ $\Bigg\}$ $\underbrace{\left(\int \rho^{\alpha(\beta-2-\delta)-\beta} |u_-|^{2\alpha}\right)^{1/2\alpha}}_{= \|u_-\|_{2\alpha,\alpha,\delta}} \leq C_\delta$

$\hookrightarrow$ Moser iteration $\Rightarrow |u_-| \leq C_{\text{unif}}$ #

Lemma (Yau's C^2-estimate) $\exists\, C_1, C_2 > 0$ independant of $\text{supp}(f)$, ε such that
$0 = \text{tr}_{\omega_0} \omega = n + \Delta_{\omega_0} u \leq C_1 \exp(C_2(u - \inf u))$

Now we consider $f = f_m$ with $\text{supp}(f_m) \nearrow M$, $\int (e^{f_m} - 1)\,\omega_0^n = 0$

Lemm 2: For each m, $\exists\, C(m)$ indep of ε s.t $|u_{m,\varepsilon}| \leq C(m)$

pf. Claim 1: $\int (e^{f_m + \varepsilon u_{m,\varepsilon}} - 1)\,\omega_0^n = 0$

Assumption: $\int (e^{f_m} - 1)\,\omega_0^n = 0$ $\Bigg\}$ $\Rightarrow \int e^{f_m}(e^{u_{m,\varepsilon}} - 1)\,\omega_0^n = 0$

$\Rightarrow \inf u < 0 < \sup u$

Claim: $\exists\, x_{max}, x_{min} \in \text{supp}(f_m)$ s.t. $u(x_{max}) = \sup u$, $u(x_{min}) = \inf u$

If not either ① $\sup u$ was attained at $x_{max} \notin \text{supp}(f_m)$

or ② $\exists \{x_k\}_k$, $u(x_k) \to \sup u$, $x_k \to \infty$

① at x_{max} $\underbrace{(\omega_0 + dd^c u)^n (x_{max})}_{\geq\, \omega_0^n(x_{max})} = e^{\varepsilon u(x_{max})}\,\omega_0^n(x_{max}) > \omega_0^n(x_{max})$ contradiction

② Similar argument + Yau's max prin

Same for inf. #

If $u_{\psi_\delta} \geq 0$, from Lemma 1: $-C \leq u \leq u_{\psi_\delta} + C$

Set $v = -\min\{u - \inf u - 1, 0\} \Rightarrow \begin{cases} v(x_{min}) = 1 \\ 0 \leq v \leq 1 \end{cases}$

$B_r(x_{min}) =: B_r$, $G(x,y)$: Dirichlet Green's function on B_1

$\eta = \begin{cases} 1 & \text{on } B_{1/2} \\ 0 & \text{outside } B_{3/4} \end{cases}$ a cutoff

Remark: Since $\text{supp}(f_m)$ is cpt, $\text{inj}(x) \geq r_m$ $\forall\, x \in \text{supp}(f_m)$. for some $r_m > 0$. Hence, we can rescale $B_{r_m}(x_{min})$ to $B_1(x_{min})$ and Green's fcns are under control there.

Since $-\inf u \leq C$, Yau's C^2: $\underbrace{n + \Delta u}_{= n - \Delta v} \leq C_1 \exp(C_2(u - \inf u)) \leq C_1' \exp(C_2 u)$ $\cdots$ ★

★ $\times\ \eta^2 \underbrace{G(x_{min}, x)}_{= G(x)} v(x)$ (note $G(x) > 0$ on $B_{3/4}$)

$\Rightarrow -\int_{B_1} \Delta v \cdot \eta^2 G \cdot v \leq \int_{B_1} \eta^2 G\, C_1' \exp(C_2 u)$ $u \leq \underbrace{\inf u}_{<1} + 1$ when $v > 0$

$\leq C \int_{B_1} \eta^2 G v$

$$-\int \Delta v \cdot \eta^2 \cdot v \cdot G = \underbrace{\int \nabla v \cdot \eta^2 v \nabla G}_{= \frac{1}{2}\int \nabla v^2 \eta^2 \nabla G} + \int |\nabla v|^2 \eta^2 G + \int \nabla v \, \nabla \eta^2 \, v \cdot G$$

Hence $\frac{1}{2}\int \nabla(\eta^2 v^2) \nabla G = \frac{1}{2}\int \nabla v^2 \eta^2 \nabla G + \frac{1}{2}\int v^2 \nabla \eta^2 \nabla G$

$$\frac{1}{2}(\eta v)^2(x_{min}) = -\int \Delta v \cdot \eta^2 v \, G - \int |\nabla v|^2 \eta^2 G \underbrace{- \frac{1}{2}\int \nabla v^2 \, \nabla \eta^2 \, G + \frac{1}{2}\int v^2 \nabla \eta^2 \nabla G}_{= \int v^2 \nabla \eta^2 \nabla G + \frac{1}{2}\int v^2 \Delta \eta^2 \, G}$$

$\| \\ 1/2$

$$\leq C \int_{B_1} v \eta^2 G + C \cdot \int_{B_1} v^2$$

$\llcorner$ G, ∇G are under control on $B_{3/4} \setminus B_{1/2}$

$\cup$

$\text{supp}(\nabla \eta)$, $\text{supp}(\Delta \eta)$

Hölder $\Rightarrow$ $$1 \leq C \left(\underbrace{\left(\int_{B_1} |v|^p\right)^{1/p}}_{\leq \text{vol}(\text{supp}(v) \cap B_1)} \left(\int_{B_1} |G|^{p'}\right)^{1/p'} + \underbrace{\int_{B_1} v^2}_{\leq \text{vol}(\text{supp}(v) \cap B_1)} \right) \quad \text{since } |v| \leq 1$$

$\Rightarrow \text{vol}(\text{supp}(v) \cap B_1) \geq 1/C$.

Then $$\left(\int \psi_\delta \omega_0^n\right) u_{\psi_\delta} = \int_M \psi_\delta u \, \omega_0^n = \underbrace{\int_{M \setminus B_1 \cap \text{supp}(v)} \psi_\delta u}_{(I)} + \underbrace{\int_{B_1 \cap \text{supp}(v)} \psi_\delta u}_{(II)}$$

$$(I) \leq \sup u \left(\int \psi_\delta \omega_0^n - \int_{B_1 \cap \text{supp}(v)} \psi_\delta \omega_0^n \right)$$

$$(II) \lesssim \rho(x_{min})^{-2-\delta} \times (\inf u + 1) < C$$

Recall: $u < u_{\psi_\delta} + C$ $\qquad$ $\text{vol}(B_1 \cap \text{supp}(v)) \geq 1/C$

$$\sup u \lesssim C \Big/ \int_{B_1 \cap \text{supp}(v)} \psi_\delta \omega_0^n, \qquad \int_{B_1 \cap \text{supp}(v)} \psi_\delta \omega_0^n \geq \frac{1}{C} \rho(x_{min})^{-2-\varepsilon}$$

$$\Rightarrow \sup u \leq C \cdot \rho^{2+\varepsilon}\left(\sup\{ \text{dist}(y, x_0) \mid y \in \text{supp}(f_m) \} \right)$$

#

Now for each m, we have $|u_{m,\varepsilon}| \leq C(m)$ $\rightsquigarrow$ can do C^2 + higher order est. (dep on m)

$\Rightarrow u_{m,\varepsilon} \to u_m$ & $|u_m| \leq C(m)$

Rmk: $\int_M |\nabla u_m|^2 \omega_0^n < +\infty$: $\int u_{m,\varepsilon} \cdot dd^c u_{m,\varepsilon} \wedge T = \int u_{m,\varepsilon} (e^{f_m + \varepsilon u_{m,\varepsilon}} - 1)\omega_0^n$

$$\Rightarrow \int |\nabla u_{m,\varepsilon}|^2 \omega_0^n \leq C \int |u_{m,\varepsilon}| \, |e^{f_m} - 1| \omega_0^n < +\infty$$

integration by parts formula as before.

Now, we want to show $u_m - (u_m)_{\psi_\delta} \xrightarrow[m\to\infty]{} u$: sol'n to (MA)

Only need to show $\|u_m - (u_m)_{\psi_\delta}\|_{L^\infty} \leq C_{unif}$

$u_m - (u_m)_{\psi_\delta}$ is also a sol'n to $(\omega_0 + dd^c u_m)^n = e^{f_m}\omega_0^n$, we can assume $(u_m)_{\psi_\delta} = 0$

To do Moser iteration as Lemma 1, we have

$$\int \zeta |\nabla |u|^{\frac{p}{2}}|^2 \omega_0^n = \frac{-np^2}{4(p-1)} \left\{ \int \zeta u|u|^{p-2}(e^f - 1)\omega_0^n + \int u|u|^{p-2} d\zeta \wedge d^c u \wedge T \right\}$$

To make sure RHS is finite for $\zeta = \chi(\rho/R)$ as $R \to \infty$,
we need to control $\int |\nabla\zeta_R||\nabla u|\, \omega_0^n \leq \frac{1}{R}\int_{B_{2R}\setminus B_R} |\nabla u| \longrightarrow 0$ as $R \to \infty$

Claim : $\lim_{R\to\infty} \frac{1}{R}\int_{B_{2R}\setminus B_R} |\nabla u|\, \omega_0^n = 0$

indeed, $\int_{B_{2R}\setminus B_R} |\nabla u|\, \omega_0^n \leq \underbrace{\left(\int_{B_{2R}\setminus B_R} |\nabla u|^2\right)^{1/2}}_{= a_R \xrightarrow[R\to\infty]{} 0} \mathrm{vol}(B_{2R}\setminus B_R)^{1/2}$

$\mathrm{vol}(B_{2R}\setminus B_R)^{1/2} \sim R^{\beta/2}$

$\hookrightarrow \frac{1}{R}\int_{B_{2R}\setminus B_R} |\nabla u| \leq R^{\beta/2 - 1} a_R \xrightarrow[R\to\infty]{} 0$ for $\beta \leq 2$ #