Métriques G_2 ALC (D'après Foscolo-Haskins-Nordström)

G_2-manifolds

The octonions $\mathbb{O}$ is the set of pairs of quaternions $(a,b) \in \mathbb{H}\times\mathbb{H}$ with product:

$$(a,b)\cdot(c,d) := (ac - \bar{d}b,\ da + b\bar{c}).$$

The product is neither commutative nor associative. Defining $\bar{x} := (\bar{a}, -b)$ for $x = (a,b) \in \mathbb{O}$, we define

$$\mathrm{Im}(x) = \frac{1}{2}(x - \bar{x}), \quad \mathrm{Re}(x) = \frac{1}{2}(x + \bar{x}).$$

There is an inner product on $\mathbb{O}$ defined by $\langle x, y\rangle = \mathrm{Re}(x\bar{y})$ corresponding to the standard inner product under the identification $\mathbb{O} = \mathbb{H}\oplus\mathbb{H} = \mathbb{R}^4 \oplus \mathbb{R}^4 = \mathbb{R}^8$.

On $\mathbb{R}^7 = \mathrm{Im}(\mathbb{O})$, we can also consider the 3-form

$$\varphi_0(x,y,z) = \langle x\cdot y, z\rangle.$$

Definition: The group G_2 is the stabilizer of φ_0 with respect to the natural action of $GL(7,\mathbb{R})$ on $\Lambda^3((\mathbb{R}^7)^*)$. It can also be seen as the group of automorphisms of $(\mathbb{O}, \cdot)$. In particular, $G_2 \subset SO(7)$.

Facts: 1) G_2 is compact simply connected of dimension 14.

2) The $GL(7,\mathbb{R})$ orbit of φ_0 is open.

Definition: A G_2-structure on an oriented 7-manifold M is a reduction of the structure group of its frame bundle to G_2. This corresponds to finding a 3-form $\varphi \in \Omega^3(M)$ such that for each $m \in M$, there is a linear oriented map $v_m : \mathbb{R}^7 \to T_m M$ such that $\varphi_0 = v_m^*(\varphi|_{\Lambda^3(T_m M)})$. Since $G_2 \subset SO(7)$, it also induces a Riemannian metric g_φ and a 4-form $*\varphi$, where $*$ is the Hodge $*$ of g_φ.

The torsion of a G_2-structure is $\nabla\varphi$, where ∇ is the Levi-Civita connection of g_φ.

Definition: A G_2-manifold is a Riemannian manifold of dimension 7 (M,g) with holonomy contained in G_2. In particular, there is an induced G_2 structure and the 3-form φ can be chosen so that $g = g_\varphi$ and $\nabla\varphi = 0$.

A G_2-manifolds is irreducible if its holonomy is precisely G_2.

Fact: G_2-manifolds are automatically Ricci-flat.

Métriques G_2 ALC (D'après Foscolo-Haskins-Nordström)

Proposition: The following are equivalent on an oriented 7-manifold equipped with G_2-structure (φ, g_φ):

1) (φ, g_φ) is torsion-free;
2) (M, g_φ) is a G_2-manifold;
3) $\nabla\varphi = 0$ on M;
4) $d\varphi = d^*\varphi = 0$
5) $d\varphi = d*\varphi = 0$.

Examples: Bryant-Salamon 1989: First complete examples (asymptotically conical cohomogeneity one examples)
Joyce 1996: First compact examples; Kummer type construction.

SU(3)-structure and Calabi-Yau 3-folds

Definition: An SU(3)-structure on a 6-dimensional manifold B is a reduction of the structure group of its frame bundle to SU(3).

Notice SU(3) can be seen as the subgroup of $GL(3,\mathbb{C})$ preserving the canonical symplectic form and holomorphic form on $\mathbb{C}^3$:

$$\omega_0 = \frac{i}{2}\sum_{k=1}^{3} dz_k \wedge d\bar{z}_k \quad , \quad \Omega_0 = dz_1 \wedge dz_2 \wedge dz_3$$

It preserves also the complexe structure of $\mathbb{C}^3$ and its canonical metric. As such, a SU(3) structure induce a Riemannian metric g_B on B, an almost complex structure J, a 2-form ω and a complex 3-form Ω. Since

$$\omega_0 \wedge \mathrm{Re}\,\Omega_0 = 0 \quad \text{and} \quad \frac{1}{6}\omega_0^3 = \frac{1}{4}\mathrm{Re}\,\Omega_0 \wedge \mathrm{Im}\,\Omega_0 ,$$

notice that the same relations holds for ω and Ω on B:

$$\omega \wedge \mathrm{Re}\,\Omega = 0 \quad , \quad \frac{1}{6}\omega^3 = \frac{1}{4}\mathrm{Re}\,\Omega \wedge \mathrm{Im}\,\Omega$$

Definition: A Calabi-Yau 3-fold is a 6-dimensional manifold B endowed with SU(3)-structure (ω, Ω) such that $d\omega = d\Omega = 0$. We say equivalently that (ω, Ω) is a torsion-free SU(3)-structure. In this case, ω and Ω are parallel with respect to the

Levi-Civita connection of the induced Riemannian metric. In particular, the complex structure is also parallel with respect to the Levi-Civita connection, so that the holonomy of the induced metric is contained in $SU(3)$.

Asymptotically locally conical (ALC) G_2-manifolds

A Riemaniannan metric g on M is ALC if outside a compact set it is asymptotic to a S^1-invariant metric,

$$g_{ALC} = \pi^* g_C + \theta^2$$

with (C, g_C) a cone, $\pi : L \to C$ a circle bundle and θ connection 1-form for L corresponding to the pull-back of a connection 1-form on $L|_{\{1\}\times S}$ where $C = [0,\infty) \times S$ and S is the link of the cone: $L = pr_2^*(L|_{\{1\}\times S})$, $\theta = pr_2^*(\theta_1)$, $pr_2 : [0,\infty)\times S \to S$ the projection on the second factor.

In particular, the volume growth of a ball of radius r in (M,g) is $O(r^{\dim M - 1})$.

Example: An ALF metric is ALC with $C = \mathbb{R}^3/\Gamma$ for a finite group $\Gamma \subset O(3)$.

The strategy of F-H-N to construct complete G_2-manifolds is to start with an Asymptotically conical (AC) Calabi-Yau 3-fold $(B, g_B, \omega_B, \Omega_B)$ and consider an S^1-invariant G_2-structure on a circle bundle

$$\begin{array}{c} S^1 \to M \\ \downarrow \\ B \end{array}$$

On M, we can consider an S^1-invariant G_2-structure of the form

$$\varphi = \theta \wedge \omega + h^{3/4} \operatorname{Re} \Omega,$$ where (ω, Ω) is a $SU(3)$ structure on B, h is a positive function on B and θ is a connection 1-form for $M \to B$.

Métriques G_2 ALC (D'après Foscolo-Haskins-Nordström)

To be torsion-free, $(\omega, \Omega, \theta, h)$ must satisfy a coupled system of nonlinear PDE (Apostolov-Salamon). This is hard to solve in general.

Idea: take the adiabatic limit where the length ε of the circle tends to zero and the metric Gromov-Hausdorff converges to a Calabi-Yau metric on B. Linearizing the rescaled Apostolov-Salamon equations at $\varepsilon = 0$ gives a coupled linear PDE for an infinitesimal deformation $(\sigma, \rho + i\hat{\rho})$ of the $SU(3)$ structure (ω_0, Ω_0), a function h and a connection 1-form θ.

Ansatz: 1) (B, ω_ε) is a fixed symplectic manifold $\Rightarrow \sigma = 0$

2) h is constant, say $h = 1$.

In this case, two of the linearized equations are

$$d\theta \wedge \omega_0^2 = 0, \qquad d\theta \wedge \mathrm{Re}\,\Omega_0 = 0 \qquad (HYM)$$

($d\theta$ is a primitive $(1,1)$-form)

Step 1: Find a solution θ of HYM $\rightsquigarrow$ ensures $\varphi_\varepsilon = \varepsilon\,\theta \wedge \omega_0 + \mathrm{Re}\,\Omega_0$ an S^1-invariant G_2-structure

Problem: $d\varphi_\varepsilon = \varepsilon\, d\theta \wedge \omega_0$ is not closed in general.

Solution: Find $\rho + i\hat{\rho}$ such that $d\rho = -d\theta \wedge \omega_0$, $d\hat{\rho} = 0$. (LAS).

$$\Rightarrow \varphi_\varepsilon^{(1)} = \varepsilon\,\theta \wedge \omega_0 + \mathrm{Re}\,\Omega_0 + \varepsilon \rho \text{ are closed.}$$

This imposes the constraint $c_1(M) \cup [\omega_0] = 0 \in H^4(B)$.

Assuming this, (LAS) can be solved formulating it as an elliptic PDE:

$$(d + d^*)\rho = * d\theta \quad \text{with } \hat{\rho} := -*\rho \text{ for } \rho \text{ a solution.}$$

This yields an approximate 1-parameter family of G_2-structures

$$\varphi_\varepsilon^{(1)} = \varepsilon\,\theta \wedge \omega_0 + \mathrm{Re}\,\Omega_0 + \varepsilon\rho \quad \text{with torsion of order } O(\varepsilon^2).$$

To construct an exact solution, the idea is then to iterate this construction to obtain a formal solution as a power series in ε and check the power series actually converges.