Utiliser cette page pour les solutions, les développements et les réponses.

Colonne réservée à la correction

§ S^1-invariant G_2-structure (torsion-free)

Let $M^7 \to B$ be a S^1-principal bundle over 6-dimensional manifold.

V the infinitesimal generator of the S^1-action.

Fact 1 Every S^1-invariant G_2-structure φ on M can be write as

$\varphi = \theta \wedge \omega + h^{3/4} \operatorname{Re} \Omega$ where (ω, Ω) is an $SU(3)$-structure

on B, $h : B \to \mathbb{R}_{>0}$ and θ a connection 1-form of $M \to B$

$$\rightsquigarrow \begin{cases} *_\varphi \varphi = -h^{1/4} \theta \wedge \operatorname{Im} \Omega + \frac{1}{2} h \omega^2 \\ g_\varphi = \sqrt{h}\, g_B + h^{-1} \theta^2 \end{cases}$$

where g_B is the metric induce by the $SU(3)$-structure (ω, Ω).

Let's look at this expression pointwise.

$p \in M$, $(e_1, \underbrace{e_2, e_3, e_5, e_4, e_7, e_6}_{(z_1, z_2, z_3)})$ basis of T_pM so that

$$\varphi_0 = e^{123} - e^{145} - e^{167} + e^{257} - e^{347} - e^{356} - e^{246}$$

$$= e^1 (e^{23} - e^{45} - e^{67}) + e^{257} - e^{347} - e^{356} - e^{246}$$

$$= \theta \wedge \omega + \operatorname{Re} \Omega \quad \text{where} \quad \begin{cases} \omega := e^{23} - e^{45} - e^{67} \\ \Omega := z^{123} \end{cases}$$

$$g_{\varphi_0} = \sum_{i=2}^{7} e^i \otimes e^i + e^1 \otimes e^1 \quad \text{and}$$

$$*_{\varphi_0}\varphi_0 = e^{4567} - e^{2367} - e^{2345} - e^{1357} + e^{1356} + e^{1346} - e^{1256} - e^{1247}$$

$$\Rightarrow *_{\varphi_0}\varphi_0 = -e^1\left(e^{357} - e^{356} + e^{256} + e^{247}\right) + e^{4567} - e^{2367} - e^{2345}$$

$$= -\theta \wedge \operatorname{Im}\Omega + \frac{1}{2}\omega^2$$

Consider a S^1-invariant function $f : M \longrightarrow \mathbb{R}_{>0}$.

$(f^{-1/2}\varphi|e_1, f^{1/4}\varphi|e_2, \dots, f^{1/4}\varphi|e_7)$ is a basis of T_pM in which

we have.

$$\begin{cases} \varphi = \theta \wedge \omega + f^{3/4}\operatorname{Re}\Omega \\ *\varphi = -f^{1/4}\theta \wedge \operatorname{Im}\Omega + \frac{1}{2} f \omega^2 \\ g_\varphi = f^{1/2} g_B + f^{-1}\theta^2 \end{cases}$$

Note that : $\omega \wedge \operatorname{Re}\Omega = 0$ and $\frac{1}{6}\omega^3 = \frac{1}{4}\operatorname{Re}\Omega \wedge \operatorname{Im}\Omega$.

Note 1. The factor is chosen such that, $d\varphi \Rightarrow d\omega = 0$

2. Since the G_2-structure is S^1-invariant, the forms in this expression are pullbacks (basics forms).

Lemma (Apostolov-Salamon equation) The S^1-invariant G_2-structure φ on M determined by the quadruplet $(\omega, \Omega, h, \theta)$ is torsion-free if and only if we have

$$(A.S)\quad \begin{cases} d\omega = 0 & d\left(h^{3/4} \operatorname{Re}\Omega\right) = -d\theta\wedge\omega \\ d\left(h^{1/4} \operatorname{Im}\Omega\right) = 0 & \frac{1}{2} dh\wedge\omega^2 = h^{1/4} d\theta\wedge \operatorname{Im}\Omega \end{cases}$$

proof

Let us remember that, a G_2-structure φ is torsion if and only if $d\varphi = 0 = d{*\varphi}$.

We have:
$$d\varphi = -\theta\wedge d\omega + d\theta\wedge\omega + d\left(h^{3/4}\operatorname{Re}\Omega\right)$$
$$d{*\varphi} = \theta\wedge d\left(h^{1/4}\operatorname{Im}\Omega\right) - h^{1/4} d\theta\wedge\operatorname{Im}\Omega + \frac{1}{2} d\left(h\omega^2\right) \qquad (*)$$

~~...~~ Futhermore.

$$\mathcal{L}_v\varphi = d(v\lrcorner\varphi) + v\lrcorner d\varphi$$
$$= d\omega + v\lrcorner d\varphi \Rightarrow d\omega + v\lrcorner d\varphi = 0$$ because φ is S^1-invariant. Similarly,

$$\mathcal{L}_v{*\varphi} = -d\left(h^{1/4}\operatorname{Im}\Omega\right) + v\lrcorner d{*\varphi} \Rightarrow d\left(h^{1/4}\operatorname{Im}\Omega\right) = v\lrcorner d{*\varphi}.$$

Then $d\varphi = 0 = d{*\varphi} \Leftrightarrow d\omega = 0 = d\left(h^{1/4}\operatorname{Im}\Omega\right)$. and by replacing in $(*)$ we have the result.

Remark 3 φ closed implies that ω is a symplectic form on B and we have a topology condition which is $[d\theta]\cup[\omega] = c_1(M)\cup[\omega] = 0 \in H^4(B)$

$\rightsquigarrow$ interpret equation (AS) as a pair of equations for the torsion ~~for the SU(3)~~ of the SU(3)-structure (ω, Ω) and the pair (h, θ).

As a result, we have the following result which identifies the intrinsic torsion of the SU(3)-structure

Proposition 4: Let (ω, Ω) be an SU(3)-structure on B. Then there exist functions $w_1, \hat{w}_1$, primitive (1,1)-forms $w_2, \hat{w}_2$, a 3-form $w_3 \in \Omega^3_{12}(B)$ and 1-forms w_4, w_5 on B identified with the intrinsic torsion such that

$$d\omega = 2w_1 \operatorname{Re}\Omega + 3\hat{w}_1 \operatorname{Im}\Omega + w_3 + w_4 \wedge \omega$$

$$d\operatorname{Re}\Omega = 2\hat{w}_1 \omega^2 + w_5 \wedge \operatorname{Re}\Omega + w_2 \wedge \omega$$

$$d\operatorname{Im}\Omega = -2w_1 \omega^2 + w_5 \wedge \operatorname{Im}\Omega + \hat{w}_2 \wedge \omega.$$

~~What~~ with this proposition admit, we have.

Lemma 5 The SU(3)-structure (ω, Ω) arising from a solution $(\omega, \Omega, h, \theta)$ of (AS) has torsion

(T): $w_1 = \hat{w}_1 = \hat{w}_2 = w_3 = w_4 = 0$, $w_5 = -\frac{1}{4} h^{-1} dh$, $w_2 = -h^{-3/4} \kappa_0$

where κ_0 is the projection of $d\theta$ onto space of primitive (1,1)-forms.

Moreover, (h, θ) satisfies
$$d\left(\frac{4}{3} h^{3/4}\right) = * \left(d\theta \wedge \mathrm{Re}\,\Omega\right) \quad , \quad d\theta \wedge \omega^2 = 0$$

which is equivalent to $d\theta = -\frac{1}{2} h^{-1/4} (J\nabla h) \lrcorner \mathrm{Re}\,\Omega + \kappa_0$.

Conversely, if (ω, Ω) is an $SU(3)$-structure on B whose torsion satisfies (T) and $d\theta$ has the previous form for some connection θ on a principal circle bundle M over B. Then $(h, \theta, \omega, \Omega)$ is a solution to (AS) coming from an S^1-invariant torsion-free G_2 structure on M.

proof.

Simply rewrite equation (AS), apply the previous proposition and then perform the identification to obtain (T).

$d\omega = 0$ (by (AS)) and $\omega \wedge \mathrm{Re}\,\Omega$ ($SU(3)$-structure equation) $\Rightarrow \omega \wedge d\mathrm{Re}\,\Omega = 0$

$\Rightarrow d\theta \wedge \omega^2$ by using the second equation in the first line of (AS).

§ <u>Adiabatic limit of S^1-invariant torsion-free G_2-structures.</u>

φ_ε be a family of S^1-invariant torsion-free G_2-structures on $M \to B$

a principal circle bundle with circle fibre shrinking to zero length as $\varepsilon \to 0$.

This is determined by

$$\varphi_\varepsilon = \varepsilon \theta_\varepsilon \wedge \omega_\varepsilon + h_\varepsilon^{3/4} \operatorname{Re} \Omega_\varepsilon \quad \rightsquigarrow \quad g_{\varphi_\varepsilon} = h_\varepsilon^{1/2} g_\varepsilon + \varepsilon^2 h_\varepsilon^{-1} \theta_\varepsilon^2$$ metric on M

induced by $(\omega_\varepsilon, \Omega_\varepsilon)$.

The equation (AS) with the torsion of the $su(3)$-structure gives

$$(AS_\varepsilon) \begin{cases} d\omega_\varepsilon = 0 \quad , \quad \frac{1}{2} dh_\varepsilon \wedge \omega_\varepsilon^2 = \varepsilon h_\varepsilon^{1/4} d\theta_\varepsilon \wedge \operatorname{Im} \Omega_\varepsilon \quad , \quad \varepsilon\, d\theta_\varepsilon \wedge \omega_\varepsilon^2 = 0 \\ d \operatorname{Re} \Omega_\varepsilon = -\frac{3}{4} h_\varepsilon^{-1} dh_\varepsilon \wedge \operatorname{Re} \Omega_\varepsilon - \varepsilon h^{-3/4} d\theta_\varepsilon \wedge \omega_\varepsilon \quad , \quad d \operatorname{Im} \Omega_\varepsilon = -\frac{1}{4} h_\varepsilon^{-1} dh_\varepsilon^{-1} \wedge \operatorname{Im} \Omega_\varepsilon . \end{cases}$$

$h_0 := \lim_{\varepsilon \to 0} h_\varepsilon$ we obtain h_0 to be constant $\boxed{\text{Assume } h_0 \equiv 1}$

Looking at the second line of (AS_ε), we have.

$\varepsilon \to 0$, $d \operatorname{Re} \Omega_0 = 0 = d \operatorname{Im} \Omega_0 \implies (\omega_0, \Omega_0)$ Calabi-Yau structure on B.

$\to$ Find a better approximation of $(h_\varepsilon, \omega_\varepsilon, \Omega_\varepsilon)$ by linearizing (AS_ε)

Take (B, ω_0, Ω_0) a Calabi-Yau 3-fold

$$\begin{cases} h_\varepsilon = 1+\varepsilon h + O(\varepsilon^2) , & \varepsilon\theta_\varepsilon = \varepsilon\theta + O(\varepsilon^2) \\ \omega_\varepsilon = \omega_0 + \varepsilon\sigma + O(\varepsilon^2) , & \Omega_\varepsilon = \Omega_0 + \varepsilon(\rho + \sqrt{-1}\,\hat\rho) + O(\varepsilon^2) \end{cases}$$

- $d\omega_\varepsilon = d\omega_0 + \varepsilon\, d\sigma \Rightarrow$ $d\omega_\varepsilon = 0$ we have $\boxed{d\sigma = 0}$

- $\frac{1}{2} dh_\varepsilon \wedge \omega_\varepsilon^2 = \varepsilon h^{1/4} d\theta_\varepsilon \wedge \mathrm{Im}\,\Omega_\varepsilon \Rightarrow \frac{1}{2}\left[\varepsilon dh \wedge (\omega_0^2 + \varepsilon^2\sigma^2 + 2\varepsilon\omega_0\sigma)\right] = \varepsilon\left[(1+\frac{1}{4}\varepsilon dh)\cdot d\theta \wedge [\mathrm{Im}\,\Omega_0 + \varepsilon\hat\rho]\right]$

$$\Rightarrow \boxed{\frac{1}{2} dh \wedge \omega_0^2 = d\theta \wedge \mathrm{Im}\,\Omega_0}$$

- $\varepsilon\, d\theta_\varepsilon \wedge \omega_\varepsilon^2 = 0 \Rightarrow \varepsilon^2 d\theta \wedge (\omega_0^2 + \varepsilon^2\sigma^2 + 2\varepsilon\, \omega_0 \wedge \sigma) = 0 \Rightarrow \boxed{d\theta \wedge \omega_0^2 = 0}$

- $d\,\mathrm{Im}\,\Omega_0 = -\frac{1}{4} h_\varepsilon^{-1} dh_\varepsilon \wedge \mathrm{Im}\,\Omega_\varepsilon$

$$\Rightarrow \varepsilon\, d\hat\rho = -\frac{1}{4}(1-\varepsilon h)\cdot\varepsilon\, dh \wedge (\mathrm{Im}\,\Omega_0 + \varepsilon\hat\rho) \Rightarrow \boxed{d\hat\rho = -\frac{1}{4} dh \wedge \mathrm{Im}\,\Omega_0}$$

- $d\,\mathrm{Re}\,\Omega_\varepsilon = -\frac{3}{4} h_\varepsilon^{-1} dh_\varepsilon \wedge \mathrm{Re}\,\Omega_\varepsilon - \varepsilon h_\varepsilon^{-1/4} d\theta_\varepsilon \wedge \omega_\varepsilon$

$$\Rightarrow \varepsilon\, d\rho = -\frac{3}{4}(1-\varepsilon h)\,\varepsilon\, dh \wedge (\mathrm{Re}\,\Omega_0 + \varepsilon\rho) - \varepsilon\left(1 - \frac{3}{4}\varepsilon h\right) d\theta \wedge (\omega_0 + \varepsilon\sigma)$$

$$\Rightarrow \boxed{d\rho = -\frac{3}{4} dh \wedge \mathrm{Re}\,\Omega_0 - d\theta \wedge \omega_0}$$

By using the constraints of the $SU(3)$-structure

$$\begin{cases} \omega \wedge \Omega = 0 \\ \frac{1}{3}\omega^3 = \frac{1}{4}\mathrm{Re}\,\Omega \wedge \mathrm{Im}\,\Omega \end{cases}$$

we obtaint. $\omega_\varepsilon \wedge \Omega_\varepsilon = 0 \Rightarrow (\omega_0 + \varepsilon\sigma) \wedge (\Omega_0 + \varepsilon(\rho + \sqrt{-1}\hat{\rho})) = 0$

$$\Rightarrow \boxed{\omega_0 \wedge (\rho + \sqrt{-1}\hat{\rho}) + \sigma \wedge \Omega_0 = 0}$$

$$\frac{1}{6}\omega_\varepsilon^3 = \frac{1}{4}\mathrm{Re}\,\Omega_\varepsilon \wedge \mathrm{Im}\,\Omega_\varepsilon \Rightarrow \frac{1}{6}(\omega_0^3 + 3\varepsilon^2\omega_0 \wedge \sigma^2 + 3\varepsilon\,\omega_0 \wedge \sigma + \varepsilon^3\sigma^3) = \frac{1}{4}(\mathrm{Re}\,\Omega_0 + \rho) \wedge (\mathrm{Im}\,\Omega + \hat{\rho})$$

$$\Rightarrow \boxed{\mathrm{Re}\,\Omega_0 \wedge \hat{\rho} + \rho \wedge \mathrm{Im}\,\Omega_0 = 2\sigma \wedge \omega_0^2}$$

Then ignoring term, of order ε^2 in (AS_ε) implies that

$$(LAS): \begin{cases} d\sigma = 0 \qquad \frac{1}{2}dh \wedge \omega_0^2 = d\theta \wedge \mathrm{Im}\,\Omega_0 \qquad d\theta \wedge \omega_0^2 = 0 \\ d\rho = -\frac{3}{4}dh \wedge \mathrm{Re}\,\Omega_0 - d\theta \wedge \omega_0 \qquad d\hat{\rho} = -\frac{1}{4}dh \wedge \mathrm{Im}\,\Omega_0 \\ \omega_0 \wedge (\rho + \sqrt{-1}\hat{\rho}) + \sigma \wedge \Omega_0 = 0 \qquad \mathrm{Re}\,\Omega_0 \wedge \hat{\rho} + \rho \wedge \mathrm{Im}\,\Omega_0 = 2\sigma \wedge \omega_0^2 \end{cases}$$

<u>Note</u> if $(h, \theta, \sigma, \rho)$ is a solution of (LAS) then

$\frac{1}{2}dh \wedge \omega_0^2 = d\theta \wedge \mathrm{Im}\,\Omega_0 \iff dh = *(d\theta \wedge \mathrm{Re}\,\Omega_0)$.

We know that $d\,\mathrm{Re}\,\Omega_0 = 0$, then $d^*h = 0 \Rightarrow h$ is harmonic

By example, B complete and h bounded implies h constant"

Thus, solving (AS_ε) to order 1 in ε is equivalent to solving (LAS).

<u>Assumption</u> $\rightsquigarrow$ Reduces (h, θ) to HYM connection

$\rightsquigarrow$ look solution of (LAS) with $\sigma = 0 = h$.

It therefore comes down to seeking a connection form θ and a 3-form ρ on B such that

$$\begin{cases} d\theta \wedge \operatorname{Im}\Omega_0 = 0 \qquad d\theta \wedge \omega_0^2 \qquad d\rho = -d\theta \wedge \omega_0 \ , \ d\hat{\rho} = 0 \\ \omega_0 \wedge (\rho + \sqrt{-1}\hat{\rho}) = 0 \ , \qquad \operatorname{Re}\Omega_0 \wedge \hat{\rho} + \rho \wedge \operatorname{Im}\Omega_0 = 0 \end{cases}$$

Thus, all the solutions (θ, ρ) form a ~~family~~ 1-parameter family

$$\varphi_\varepsilon^{(1)} = \varepsilon\theta \wedge \omega_0 + \operatorname{Re}\Omega_0 + \varepsilon\rho \quad \text{on } M.$$

$\Rightarrow d\varphi_\varepsilon^{(1)} = \varepsilon\,(d\theta \wedge \omega_0 + d\rho) = 0$, the rest of the torsion is of order $O(\varepsilon^2)$

The next step will therefore consist of:

$\rightsquigarrow$ Show that for $\varepsilon \ll 1$ any such approximation can be approximated by a solution to the equation (AS_ε).

$\rightsquigarrow$ Construct a solution of (AS) as a formal power series in ε by solving iteratively (AS_ε).