Section 4. Three-dimensional Calabi-Yau cones

Goal: understand DOs such as $\not{D}$, $d+d^*$, $\Delta = dd^* + d^*d$ on AC Calabi-Yau

$\approx$ ——— on CY cone $C(\Sigma)$

$\rightsquigarrow$ prop on Σ: Sasaki-Einstein 5-manifold

§ Sasaki-Einstein 5-manifolds

* differential form point of view

SE structure: special type of $SU(2)$-structure

4-tuple $(\eta, \omega_1, \omega_2, \omega_3)$

η: nowhere vanishing 1-form $\rightsquigarrow \mathcal{H} := \ker\eta$: codim 1 distribution

$\omega_1, \omega_2, \omega_3$ 2-forms st. $\eta\wedge\omega_i^2 \neq 0$ & $\omega_i\wedge\omega_j = \delta_{ij}\omega_1^2$

(oriented v.s V: 4-dim'l $\rightsquigarrow$ 3 dim'l sp Λ^+V^* self-dual 2-forms)

$\rightsquigarrow \omega_1, \omega_2, \omega_3$ oriented basis of the subsp $\subset \Lambda^2\mathcal{H}^*$ they span

$SU(2) \subset SO(4) \subset SO(5) \rightsquigarrow$ Riem metric g_Σ

The $SU(2)$-structure $(\eta, \omega_1, \omega_2, \omega_3)$ is called Sasaki-Einstein if

$d\eta = 2\omega_1$, $d\omega_2 = -3\eta\wedge\omega_3$, $d\omega_3 = 3\eta\wedge\omega_2$

$\Updownarrow$

Conical $SU(3)$ structure on $C(\Sigma)$ by $\omega_C = r\,dr\wedge\eta + r^2\omega_1$

$\Omega_C = r^2(dr + ir\eta)\wedge(\omega_2 + i\omega_3)$

is torsion free: $d\omega_C = 0$

$d\Omega_C = 0$

$$d\omega_C = -r\,dr\wedge d\eta + 2r\,dr\wedge\omega_1 + r^2 d\omega_1$$
$$= 0 \Leftrightarrow d\omega_1 = 0,\ d\eta = 2\omega_1$$

$$d\Omega_C = 2r\,dr\wedge ir\eta\wedge(\omega_2 + i\omega_3) + r^2\, i\,dr\wedge\eta\wedge(\omega_2 + i\omega_3)$$
$$+ ir^3 d\eta\wedge(\omega_2 + i\omega_3)$$
$$- r^2(dr + ir\eta)\wedge(d\omega_2 + i\,d\omega_3)$$
$$= i3r^2\,dr\wedge\eta\wedge(\omega_2 + i\omega_3) - r^2 dr\wedge(d\omega_2 + i\,d\omega_3) + ir^3 d\eta\wedge(\omega_2 + i\omega_3)$$

Rmk: $g_C = dr^2 + r^2 g_\Sigma$, o.n.b. $E_i = \frac{1}{r} e_i$ $i = 1, \dots, 5$, $E_0 = \partial_r$

$\mathrm{Ric}_{g_C}(E_i, E_j) = \mathrm{Ric}_{g_\Sigma}(E_i, E_j) - (n-1)\delta_{ij}$

$= 0 \Leftrightarrow \mathrm{Ric}_{g_\Sigma} = (n-1) g_\Sigma \Rightarrow \mathrm{Scal}(g_\Sigma) = n(n-1)$

$\hookrightarrow$ our case $\mathrm{Scal}(g_\Sigma) = 5 \cdot 4 = 20$.

Consider ξ dual to η wrt g_Σ

$\hookrightarrow$ Reeb vector field. $\quad \underset{\hookrightarrow \text{ unit length}}{\overset{\text{normalization}}{\eta(\xi) = 1}}$, $\iota_\xi d\eta = 0$

Rmk: ξ unit length killing: $\mathcal{L}_\xi \omega_1 = 0$, $\mathcal{L}_\xi \eta = 0$

$g_\Sigma = \eta^2 + g_H$, $g_H = \omega_1(\cdot, J_1 \cdot)$

$T\Sigma = \mathbb{R}\xi \oplus \mathcal{H}$

$\hookrightarrow$ inherit almost hermitian structures by $g_\Sigma|_\mathcal{H}$ and $(\omega_1, \omega_2, \omega_3)|_\mathcal{H}$

$\rightsquigarrow$ transversal almost complex structures J_1, J_2, J_3

$\left(J_i J_j = -J_j J_i = J_k \text{ for } (ijk) \text{ cyclic permutation of } (123) \right)$

$\hookrightarrow$ extend J_i to $T\Sigma$ by $J_i \xi = 0$

Lem 4.3 (lemma of Hodge $*$ & decomposition of sp of forms)

Let Σ^5 endow with an $SU(2)$ structure $(\eta, \omega_1, \omega_2, \omega_3)$

(1) volume form of g_Σ induced by $SU(2)$ struc $= dv_\Sigma = \frac{1}{2}\eta \wedge \omega_1^2$

(2) $*\eta = \frac{1}{2}\omega_1^2$ and $*\gamma = -J_i \gamma \wedge \eta \wedge \omega_i$ for every $\gamma \in \mathcal{H}^*$ & $i = 1, 2, 3$

(3) $\Lambda^2 T^*\Sigma = \mathbb{R}\omega_1 \oplus \mathbb{R}\omega_2 \oplus \mathbb{R}\omega_3 \oplus \Lambda_0^{1,1}\mathcal{H}^* \oplus \mathcal{H}^* \wedge \eta$

where $\Lambda_0^{1,1}\mathcal{H}^* =$ primitive 2-forms on $\mathcal{H}$ wrt (ω_1, J_1)

(4) For $i = 1, 2, 3$, $\quad *\omega_i = \eta \wedge \omega_i$

$*(\gamma \wedge \eta) = J_i \gamma \wedge \omega_i \quad \forall \gamma \in \mathcal{H}^*$

$*\sigma = -\eta \wedge \sigma, \quad \forall \sigma \in \Lambda_0^{1,1}\mathcal{H}^*$.

§ Eigenvalue estimates of Δ

Prop 4.9: Σ^5: SE with non-constant curvature

(1) 1-st eigenvalue of scalar $\Delta > 5$

(2) 1-st eigenvalue of $\Delta \curvearrowright$ coclosed 1-forms ≥ 8

& eigenspace w/ eigenvalue $= 8$ consists of 1-forms dual to $K(\Sigma)$ space of killing fields

(3) Σ regular SE (i.e. $\Sigma / S^1_{\mathfrak{Z}}$ = Fano KE surface),

then 1-st eigenvalue of $\Delta \curvearrowright$ coclosed 2-forms > 4.

pf. (1) Lichnerowicz–Obata thm: $Ric \geq (n-1)\kappa \Rightarrow \lambda_1 \geq \kappa \cdot n$

& if $\lambda_1 = \kappa \cdot n$, then $\Sigma \simeq (S^n, g_{round})$

(2) A result of Lichnerowicz + $Scal = 20$. (check?)

(3) Assume by contradiction: $\exists\, \tau$ coclosed 2-form on Σ

st. $\Delta\tau = \mu^2 \tau$ for some $\tau \in (0, 2]$

Thm A2: $r^\mu \tau$ is a harmonic 2-form on $C(\Sigma)$

Recall that $\Lambda^2 \mathbb{R}^6 = \mathbb{R}\omega_C \oplus \{X \lrcorner \mathrm{Re}\Omega\} \oplus \Lambda^{1,1}_{prim}$ for $SU(3)$-structure

$\hookrightarrow \underbrace{r^\mu \tau}_{\text{order } \mu-2} = f \cdot \underbrace{\omega_C}_{\text{order } 0} + X \lrcorner \mathrm{Re}\Omega_C + \text{prim.}$

$\hookrightarrow f$: harmonic of order $\mu - 2 \in (-2, 0]$

Prop 4.12 $\Rightarrow f = 0$ for $\mu \in (0,2)$,

$f = Kr^0$ for $\mu = 2$.

$X^\flat$ is harmonic & homogeneous of rate $\mu - 2 \in (-2, 0]$

Prop 4.13 $\Rightarrow$ no homogeneous harmonic 1-forms of rate $\in [-4, 0]$.

$$\Rightarrow r^\mu \tau = \begin{cases} \text{primitive } (1,1) & \text{when } \mu \in (0,2) \\ \text{primitive } (1,1) + K\omega_C & \mu = 2. \end{cases}$$

$$\Rightarrow r^\mu \tau \wedge \omega_C^2 = \begin{cases} 0 & \text{when } \mu \in (0,2) \\ K\omega_C^3 & \mu = 2 \end{cases}$$

And $r^\mu \tau \wedge \underbrace{\mathrm{Re}\,\Omega}_{= r^2 dr\wedge\omega_2 - r^3\eta\wedge\omega_3} = 0$ (no $X \lrcorner \mathrm{Re}\,\Omega$ component in $r^\mu \tau$)

$\Rightarrow \tau\wedge dr\wedge\omega_2 = 0$ & $\tau\wedge\eta\wedge\omega_3 = 0$

$\Rightarrow \tau\wedge\omega_2 = \tau\wedge\omega_3 = 0$

$\tau\wedge\eta\wedge\omega_2 = \tau\wedge\eta\wedge\omega_3 = 0$

$\Lambda^2 T\Sigma = \mathbb{R}\omega_1 \oplus \mathbb{R}\omega_2 \oplus \mathbb{R}\omega_3 \oplus \Lambda_0^{1,1}\mathcal{H}^* \oplus \mathcal{H}^*\wedge\eta$

$\hookrightarrow \tau = K\omega_1 + \tau_0$, $\tau_0 \in \Lambda_0^{1,1}\mathcal{H}^*$.

$*\tau = K\eta\wedge\omega_1 - \eta\wedge\tau_0$

$$0 = d*\tau = d(K\eta\wedge\omega_1 - \eta\wedge\tau_0) = K\,2\omega_1\wedge\omega_1 - K\eta\wedge \overset{\nearrow 0}{d\omega_1} - d\eta\wedge\tau_0 + \eta\wedge d\tau_0$$

$$= 2K\omega_1^2 - \overset{\nearrow 0}{\cancel{\omega_1\wedge\tau_0}} + \eta\wedge d\tau_0$$

$\Rightarrow K=0$ & $\tau = \tau_0$ regardless of the value of $\mu \in (0,2]$.

Consider now $d\tau$:

Since $\tau\wedge\omega_i = 0 \Rightarrow \overset{0}{\underset{\shortparallel}{}} d(\tau\wedge\omega_i) = \begin{cases} d\tau\wedge\omega_1 \\ d\tau\wedge\omega_2 - \tau\wedge d\omega_2 = d\tau\wedge\omega_2 - 3\overset{0}{\overset{\shortparallel}{\overbrace{\tau\wedge\eta\wedge\omega_1}}} \\ d\tau\wedge\omega_3 + 0 \end{cases}$

$\Rightarrow d\tau\wedge\omega_i = 0 \quad \forall i = 1,2,3$

$\Rightarrow \langle d\tau, \eta\wedge\omega_i\rangle = 0$

$\shortparallel\; d\tau\wedge*(\eta\wedge\omega_i) = d\tau\wedge \iota_{\eta^\#}(*\omega_i) = d\tau\wedge \iota_3(\eta\wedge\omega_i)$

$= d\tau\wedge\omega_i = 0$

$\Lambda^3 T^*\Sigma = *\Lambda^2 T\Sigma = \mathbb{R}\eta\wedge\omega_1 \oplus \mathbb{R}\eta\wedge\omega_2 \oplus \mathbb{R}\eta\wedge\omega_3 \oplus (\eta\wedge\sigma) \oplus (\gamma\wedge\omega_1)$

$\sigma \in \Lambda_0^{1,1}\mathcal{H}^*$ $\quad \gamma \in \mathcal{H}^*$

$\Rightarrow d\tau = \gamma\wedge\omega_1 + \mu\,\eta\wedge\tau'$ for some $\gamma \in \Lambda^1\mathcal{H}^*$

$\tau' \in \Lambda_0^{1,1}\mathcal{H}^*$.

Since $\underset{\overset{\parallel}{0}}{d*\tau} = \eta\wedge d\tau = \eta\wedge\nu\wedge\omega_1 \Rightarrow \nu = 0$

$\Rightarrow d\tau = \mu\cdot\eta\wedge\tau'$

$\Delta\tau = \mu^2\tau$

$\overset{\shortparallel}{d^*}d\tau = \mu\, d^*(\eta\wedge\tau') = -\mu * d * (\eta\wedge\tau') = \mu * d\tau'$

$\Rightarrow d\tau' = *\mu\tau = -\mu\,\eta\wedge\tau$

↳ (τ,τ') satisfies 1-st order system $\begin{cases} d\tau = \mu\,\eta\wedge\tau' = -\mu*\tau \\ d\tau' = -\mu\eta\wedge\tau = \mu*\tau \end{cases}$

Consider $\tau^c := \tau + i\tau' \in \Lambda_0^{1,1}\mathcal{H}^*\otimes\mathbb{C}$

$\Delta\tau^c = \mu^2\tau^c$ & $\underline{\mathcal{L}_\xi\tau^c = -i\mu\tau^c}$

$\Leftarrow \tau,\tau' \in \Lambda_0^{1,1}\mathcal{H}^* \Rightarrow \iota_\xi\tau,\ \iota_\xi\tau' = 0$

$\Rightarrow \mathcal{L}_\xi\tau = \mu\,\iota_\xi(\eta\wedge\tau') = \mu\tau'$

Claim: $\tau^c = 0$ when Σ^5 is a regular SE.

pf. Assume Σ is simply connected (can passing to $\tilde{\Sigma}$)

$\Sigma = U(1)$-bundle of $L = K_D^{1/I}$, D: Del Pezzo

I: Fano index

↳ largest integer $\in\mathbb{N}$ s.t. $-K_D \sim k\underset{\text{ample}}{H}$

ω_1 KE on D s.t. $Ric(\omega_1) = 6\omega_1$

$d\eta = 2\omega_1$, $V = \frac{I}{3}\xi$ v.f. of period 2π generating S^1-action on $\Sigma\to D$

$\left([d\eta] = 2[\omega_1] = \frac{1}{3}[Ric(\omega)] = \frac{2\pi}{3}c_1(D) = -\frac{2\pi I}{3}c_1(L) \right.$

$\left. \eta \text{ differ from the curvature form by factor } I/3 \right)$

$\mathcal{L}_\xi\tau^c = -i\mu\tau^c \Rightarrow \mathcal{L}_V\tau^c = -i\underbrace{\left(\frac{\mu I}{3}\right)}_{= m\in\mathbb{N}}\tau^c \Rightarrow \mu = \frac{3}{I}m,\ m\in\mathbb{Z}_{>0}$

For Fano KE surf. except $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^2$, we have $I = 1 \Rightarrow \mu \geq 3$

(not the case)
$\mu \in (0, 2]$

When $D = \mathbb{P}^2 \Rightarrow I = 3$
$\quad\quad\ \ \mathbb{P}^1 \times \mathbb{P}^1 \quad\quad\ 2$

$\mathcal{L}_v \tau^c = -im\tau^c$

τ^c can be viewed as a L^m-valued ASD 2-form on D.

[2] : $\Delta \tau^c = \mu^2 \tau^c \Rightarrow \tau^c$ is a harmonic section of $\Lambda^- T^*D \otimes L^m$
with connection $A = i\frac{3m}{I}\eta$

A: self-dual connection : Weitzenböck formula for Δ_A coincides
with std Weitzenböck formula for Δ on ASD
2-form

$$\hookrightarrow \ 0 = \Delta_A \tau^c = \nabla_A^* \nabla_A \tau - 2W^-(\tau^c) + \frac{1}{3}\mathrm{Scal} \cdot \tau^c$$

$$\left(\Delta = \nabla^* \nabla + R, \quad R = \begin{cases} -2W^- + \frac{1}{3}\mathrm{Scal} & \text{on } \Lambda^2_- \\ -2W^+ + \frac{1}{3}\mathrm{Scal} & \phantom{\text{on }} \Lambda^2_+ \end{cases} \right)$$

When $D = \mathbb{P}^2$, $R > 0 \Rightarrow \tau^c = 0$

When $D = \mathbb{P}^1 \times \mathbb{P}^1$, $\Lambda^2_- T^*D = \underbrace{\mathbb{R}\,\pi_1^*\omega_{FS} \oplus \mathbb{R}\,\pi_2^*\omega_{FS}}_{R=0} \oplus \underbrace{E}_{R>0}$ $\leftarrow$ $\mathbb{C}$-l.b.

$\Rightarrow \tau^c$ must be an L^m-valued parallel multiple of ω_{FS}
but L^m is non-trivial for $m \neq 0 \Rightarrow \tau^c = 0$ #

§ Differential forms on CY cones

Defn: A k-form on $C(\Sigma)$ α is homogeneous of order λ if
$$\alpha = r^\lambda (r^{k-1} dr \wedge \alpha_{k-1} + r^k \alpha_k)$$
for some $\alpha \in \Omega^{k-1}(\Sigma)$, $\alpha_k \in \Omega^k(\Sigma)$.

Thm A2 + Prop 4.9 (1)+(2) $\Rightarrow$

Prop 4.12: Let u be a harmonic function on $C(\Sigma)$, homogeneous of order λ.
Then $u = 0$ if $\lambda \in [-5, 1] \setminus \{-4, 0\}$
and $u = K r^\lambda$ for some $K \in \mathbb{R}$, if $\lambda = -4, 0$

Pf. Thm A2 (4): $u = r^\lambda \beta$ & $\Delta \beta = (\lambda + 6 - 2)\lambda \beta$
$$= (\lambda + 4)\lambda \beta$$
$(\lambda+4)\lambda \geq 0 \Rightarrow \lambda \in]-\infty, -4] \cup [0, +\infty[$

for $\lambda \in [-5, 1]$, $u = 0$ when $\lambda \in (-4, 0)$

When $\lambda = -4, 0 \Rightarrow \Delta\beta = 0 \Rightarrow \beta = K$: cst

When $\lambda \in [-5, -4) \cup (0, 1]$,
we need $(\lambda+4)\lambda > 5$ (Prop 4.9 (1))
$\Rightarrow$ not the case $\Rightarrow$ $u = 0$ for $\lambda \in [-5, 1] \setminus \{-4, 0\}$
$u = K r^\lambda$ for $\lambda = -4, 0$. #

Prop 4.13: Let γ be a harmonic 1-form homogeneous of order λ.
Then
$$\gamma = \begin{cases} K r\,dr + d(\frac{1}{2} r^2 \alpha) + r^2 \beta, & K \in \mathbb{R}, \Delta\alpha = 12\alpha, \beta^b \in K(\Sigma), \text{ if } \lambda = 1 \\ d(\frac{1}{\lambda+1} r^{\lambda+1} \alpha), & \Delta\alpha = (\lambda+5)(\lambda+1)\alpha \quad \text{if } \lambda \in (0,1) \\ 0 & \text{if } \lambda \in [-4, 0] \\ r^\lambda \alpha\,dr - \frac{r^{\lambda+1}}{\lambda+3} d\alpha, & \Delta\alpha = (\lambda+3)(\lambda-1)\alpha \quad \text{if } \lambda \in (-5, -4) \\ K r^{-5} dr + (r^{-5}\alpha\,dr + \frac{1}{2} r^{-4} d\alpha) + r^{-4}\beta & \text{if } \lambda = -5 \\ \quad K \in \mathbb{R}, \Delta\alpha = 12\alpha, \beta^b \in K(\Sigma) & \end{cases}$$

Consider the following operators: $\quad g \mapsto \pi_1\, dd^*\left(\tfrac{1}{2} g\,\omega_0^2\right)$ $\qquad$ (4.15)

$$(f, r) \mapsto \pi_{1\oplus 6}\, d^*d\left(f\,\omega_0 + r^\# \lrcorner \operatorname{Re}\Omega_0\right)$$

where (ω_0, Ω_0) $SU(3)$-structure

$\rightsquigarrow \Lambda^2 \mathbb{R}^6 = \Lambda^2_1 \oplus \Lambda^2_6 \oplus \Lambda^2_8$, $\quad \Lambda^2_1 = \mathbb{R}\omega_0$, $\Lambda^2_6 = \{X \lrcorner \operatorname{Re}\Omega_0 : X \in \mathbb{R}^6\}$

Λ^2_8 = primitive $(1,1)$-type

Lem 2.19: (B, ω, Ω) CY 3-fold (torsion-free $SU(3)$-structure)

Then $\pi_1\, dd^*\left(\tfrac{1}{2} g\,\omega^2\right) = \tfrac{1}{3}(\Delta g)\cdot\omega^2$, $\forall g$

and $(f, r) \mapsto \pi_{1\oplus 6}\, d^*d\left(f\omega + r^\# \lrcorner \operatorname{Re}\Omega\right)$

can be identified with $\left(\tfrac{2}{3}\Delta f,\ dd^*r + \tfrac{2}{3} d^*dr\right)$.

Prop 4.16: For $\lambda \in (-4, 0)$, there are no elements homogeneous of order λ in the kernel of operators in (4.15).

pf. Lem 2.19 $\rightsquigarrow$ we need to study the existence of homogeneous fcns & 1-forms on CY cone $C(\Sigma)$ in $\ker\Delta$ & $\ker\left(\Delta - \tfrac{1}{3} d^*d\right)$ respectively.

Prop 4.12 $\Rightarrow$ no non-zero harmonic fcns on $C(\Sigma)$ homogeneous of order $\lambda \in (-4, 0)$

For $\beta = r^\lambda(\beta_0\, dr + r\beta_1)$, where $\beta_0 \in \Omega^0(\Sigma)$, $\beta_1 \in \Omega^1(\Sigma)$

$\leftarrow$ homogeneous 1-form on $C(\Sigma)$

$$\left(\Delta - \tfrac{1}{3} d^*d\right)\beta = 0$$

note: by Lem A1: $\Delta r = -\dfrac{(n-1)}{r}$, $\quad n = 6 \Rightarrow \Delta r = -\dfrac{5}{r}$

Write $\beta = r^{\lambda+1}\left(\dfrac{dr}{r}\beta_0 + \beta_1\right)$

Lem A1: $\Delta\beta = r^{\lambda+1-2}\left(\dfrac{dr}{r} \wedge A + B\right) = r^{\lambda-2}(dr \wedge A + rB)$

$$A = \Delta\beta_0 - (\lambda - 1)(\lambda + 5)\beta_0 - 2d^*\beta_1$$

$$B = \Delta\beta_1 - (\lambda + 3)(\lambda + 1)\beta_1 - 2d\beta_0$$

& we have $d^*d\beta = d^*\left(-r^\lambda dr \wedge d\beta_0 + (\lambda+1) r^\lambda dr \wedge \beta_1 + r^{\lambda+1} d\beta_1\right)$

$=$

$$(\Delta - \tfrac{1}{3} d^* d\beta) = 0 \iff \begin{cases} \tfrac{2}{3} d^* d\beta_0 = (\lambda-1)(\lambda+5)\beta_0 - \tfrac{1}{3}(\lambda-5) d^*\beta_1 \\ (dd^* + \tfrac{2}{3} d^* d)\beta_1 = \tfrac{2}{3}(\lambda+1)(\lambda+3)\beta_1 + \tfrac{1}{3}(\lambda+9) d\beta_0 \end{cases} \qquad (4.17)$$

When $\lambda \neq -5, 1$, $\int_\Sigma$ (1st eq of 4.17) $\Rightarrow \int \beta_0 = 0$

$(\lambda-5) d^*\beta_1 - (\lambda-1)(\lambda+9)\beta_0$ is an eigenfunction of eigenvalue $(\lambda+1)(\lambda+5)$

$d^*\beta_1 - (\lambda+5)\beta_0$ $\qquad (\lambda-1)(\lambda+3)$

If $\lambda \in [-4, 0] \Rightarrow (\lambda+1)(\lambda+5) \leq 5$ & $(\lambda-1)(\lambda+3) \leq 5$

Note $\langle d^*\beta_1, cst \rangle_{L^2} = 0 = \langle \beta_0, cst \rangle$ ← mean value of $\beta_0 = 0$

$\Rightarrow (\lambda-5) d^*\beta_1 - (\lambda-1)(\lambda+9)\beta_0 = 0$, $\quad d^*\beta_1 - (\lambda+5)\beta_0 = 0$

$\Rightarrow (\lambda-5) d^*\beta_1 = (\lambda-1)(\lambda+9)\beta_0$

$\qquad \| $

$\qquad \qquad \Rightarrow \beta_0 = 0 = d^*\beta_1$ unless $\lambda = -2$

$(\lambda-5)(\lambda+5)\beta_0$

If $\lambda = -2 \Rightarrow d^*\beta_1 = 3\beta_0$ & 1st eq in (4.17) $\equiv (\Delta\beta_0 + 3\beta_0 = 0)$

$\qquad \Rightarrow \beta_0 = 0 = d^*\beta_1$

↳ $\beta_0 = 0 = d^*\beta_1 \quad \forall \lambda \in [-4, 0]$

2nd eq in (4.17) $\equiv \big(d^* d\beta_1 = (\lambda+1)(\lambda+3)\beta_1 \big)$

$$d^* d\beta_1 \underset{d^*\beta_1 = 0}{=} \Delta\beta_1$$

↳ β_1 : coclosed eigenform of Δ w/ eigenvalue $(\lambda+1)(\lambda+3) < 8$ for $\lambda \in (-5, 1)$

$\Rightarrow (\beta_0, \beta_1) = (0, 0)$ & hence $\beta = 0$ #

§ Closed & coclosed forms.

Want to understand indicial root of $d+d^*$.

Thm A2 + $b_1(\Sigma)=0$ + Prop 4.9 $\Rightarrow$ Prop 4.20 & Prop 4.22

Prop 4.20: Let γ be a harmonic 2-form on $C(\Sigma)$ homogeneous of order λ

Decompose $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ as in Thm A2.

(1) $\gamma_1 = r^{\lambda+1} dr \wedge df$, where $f \in \Omega^0(\Sigma)$ s.t. $\Delta f = \lambda(\lambda+4) f$.

In particular, $\gamma_1 = 0 \quad \forall \lambda \in [-5,1]$

(2) $\gamma_2 = 0 \quad \forall \lambda \in (-6,0)$.

If $\lambda = -6$ or 0, $\gamma_2 = d\left(\frac{1}{\lambda+2} r^{\lambda+2} \alpha\right)$ where $\alpha^b \in K(\Sigma)$

(3) $\gamma_3 = 0 \quad \forall \lambda \in (-4,2)$.

If $\lambda = -4$ or 2, $\gamma_3 = r^{\lambda+1} dr \wedge \alpha - \frac{1}{\lambda+2} r^{\lambda+2} d\alpha$, where $\alpha^b \in K(\Sigma)$

(4) $\gamma_4 = r^{\lambda+2}\beta$ for a coclosed 2-form β on Σ s.t. $\Delta\beta = (\lambda+2)^2\beta$

In particular, $\gamma_4 = 0 \quad \forall \lambda \in [-4,0] \setminus \{-2\}$

Prop 4.22. Let γ be a harmonic 3-form on $C(\Sigma)$ homogeneous of order λ.

Decompose $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$

(1) $\gamma_1 = 0 \quad \forall \lambda \in (-5,1) \setminus \{-3,-1\}$.

harmonic 2-form

If $\lambda = -1$ or -3, $\gamma_1 = r^{\lambda+2} dr \wedge \alpha$, where $\Delta\alpha = 0$

$\lambda = -5$ or 1, $\gamma_1 = r^{\lambda+2} dr \wedge d\beta$, $\beta^b \in K(\Sigma)$.

(2) $\gamma_2 = d\left(\frac{1}{\lambda+3} r^{\lambda+3} \alpha\right)$, where α : coexact 2-form on Σ, $\Delta\alpha = (\lambda+3)^2\alpha$

In particular, if Σ is regular, $\gamma_2 = 0 \quad \forall \lambda \in [-5,-1]$

(3) $\gamma_3 = r^{\lambda+2} dr \wedge \alpha - \frac{1}{\lambda+1} d\alpha$, where α is coexact 2-form on Σ,

$\Delta\alpha = (\lambda+1)^2\alpha$

In particular, if Σ is regular, $\gamma_3 = 0 \quad \forall \lambda \in [-3,1]$

(4) $*\gamma_4$ satisfies the same condition as γ_1.

Closed & coclosed even degree forms of rate -2

odd ——— rate -3

Prop 4.24: Let γ be a closed & coclosed form of even deg on $C(\Sigma)$ homogeneous of rate $\lambda = -2$.

Then γ has only components of pure deg 2 and 4, both are closed & coclosed:

$$\gamma = \tau_1 + r\, dr \wedge \eta \wedge \tau_2$$

for harmonic 2-forms τ_1, τ_2 on Σ.

Moreover if Σ is regular SE, then there are no closed & coclosed even degree forms on $C(\Sigma)$ homogeneous of rate $\lambda \in (-4,0) \setminus \{-2\}$.

pf. Write $\gamma = \gamma_0 + \gamma_2 + \gamma_4 + \gamma_6$, with $\gamma_k \in \Omega^k$

Prop 4.12 $\Rightarrow \gamma_0 = \gamma_6 = 0 \quad \forall \lambda \in (-4,0)$

Write $\gamma_k = r^\lambda (r^{k-1} dr \wedge \alpha_{k-1} + r^k \beta_k)$

$d\gamma = 0 = d^*\gamma$

$$\Rightarrow \quad d^*\alpha_1 = 0 \qquad\qquad d^*\beta_2 - (\lambda+4)\alpha_1 = 0$$

$$d\alpha_1 + d^*\alpha_3 - (\lambda+2)\beta_2 = 0 \qquad d\beta_2 + d^*\beta_4 - (\lambda+2)\alpha_3 = 0$$

$$d\alpha_3 - (\lambda+4)\beta_4 = 0 \qquad\qquad d\beta_4 = 0$$

If $\lambda = -2$

$d\alpha_1 = -d^*\alpha_3 \Rightarrow \|d\alpha_1\|^2_{L^2} = -\langle d\alpha_1, d^*\alpha_3\rangle_{L^2} = -\langle d^2\alpha_1, \alpha_3\rangle_{L^2} = 0$

$\Rightarrow d\alpha_1 = 0$, & similarly $d^*\beta_4 = 0$

$\Rightarrow \alpha_1$ & β_4 are both closed & coclosed.

But $H^1(\Sigma) = 0 \simeq H^4(\Sigma) \Rightarrow \alpha_1 = 0 = \beta_4$

$\Rightarrow d^*\alpha_3 = 0 = d\alpha_3 = d^*\beta_2 = d\beta_2 \Rightarrow \alpha_3$ & β_2 are both closed & coclosed

$\Rightarrow \gamma_2 = \beta_2, \quad \gamma_4 = r\, dr \wedge \alpha_3$

Prop 4.20 $\Rightarrow$ no harmonic forms of deg 2 & 4 of rate $\lambda \in (-4,0) \setminus \{-2\}$ #

Prop 4.25: There are no closed & coclosed 2-forms on $C(\Sigma)$ homogeneous of rate $\lambda \in (-6,0) \setminus \{-2\}$

no need to assume Σ is regular SE.

pf Rmk A5: if γ is closed & coclosed $\Rightarrow \gamma = \gamma_1 + \gamma_2 + \gamma_4$

$\gamma_1 = 0$ since $b_1(\Sigma) = 0$

$\gamma_4 = 0$ unless $\lambda = -2$ (?)

Prop 4.20 (2) $\Rightarrow \gamma_2 = 0 \ \forall \lambda \in (-6,0)$ #

Prop 4.26: Let γ be a closed & coclosed form of odd degree on $C(\Sigma)$ homogeneous of rate $\lambda \in [-4,0]$. Then γ is of pure deg 3.

Moreover, if $\lambda = -3$ then $\gamma = \eta \wedge \tau_1 + \frac{dr}{r} \wedge \tau_2$ for harmonic 2-forms τ_1, τ_2 on Σ,

and if $\lambda \in [-4,0] \setminus \{-3\}$, then $\gamma = d\left(\frac{r^{\lambda+3}}{\lambda+3} \alpha\right)$,

where α is a coclosed 2-form on Σ satisfying $\Delta \alpha = (\lambda+3)^2 \alpha$

In particular, if Σ is regular SE, then there are no closed & coclosed odd deg forms on $C(\Sigma)$ homogeneous of degree $\lambda \in [-4,-1] \setminus \{-3\}$