Section 5. Differential forms on AC Calabi–Yau 3-folds

Let (B, ω_0, Ω_0) be an AC Calabi–Yau 3-fold.

Goal: understand closed & coclosed forms on B of decay at ∞

§ L^2-cohomology

$L^2\mathcal{H}^k(B) := \{\sigma \in \Omega^k \cap L^2 \mid d\sigma = 0 = d^*\sigma\}$ L^2 closed & coclosed forms

View B as a mfd with bdry Σ.

relative cohomology ⋯ long exact seq of cohomology $\cdots \to H^{k-1}(B) \to H^{k-1}(\Sigma) \to H^k_c(B) \to H^k(B) \to H^k(\Sigma)$ ⋯

Thm 5.2. Let $(B, \omega_0 \Omega_0)$ be an AC CY 3-fold. We have

$$L^2\mathcal{H}^k(B) \simeq \begin{cases} H^k_c(B) & \text{if } k = 0, 1, 2 \\ \operatorname{im}(H^3_c(B) \to H^3(B)) & k = 3 \\ H^k(B) & k = 4, 5, 6 \end{cases}$$

Rmk (Indicial root): $(C(\Sigma), g_c) = ((R, \infty) \times \Sigma,\ dr^2 + r^2 g_\Sigma) \overset{\text{asymp}}{\simeq} (B \setminus K, g_{AC})$ on the link Σ

For an elliptic DO L, we want to look for model sol'n $u(r, \theta) = r^\lambda \phi(\theta)$

L_C = leading part of L at ∞. (cone operator)

$L_C(r^\lambda \phi) = r^{\lambda-2}(I(\lambda)\phi)$ $I(\lambda)$ an operator on Σ.

Indicial root = value of λ for which $\exists$ a nonzero ϕ with $I(\lambda)\phi = 0$.

Fact: ① If ν weight avoids all indicial roots, $L: C^{k+2,\alpha}_{\nu+2} \to C^{k,\alpha}_{\nu}$ is Fredholm.

② dim of ker L, coker L doesn't jump, when ν doesn't cross indicial root.

Note that $\|u\|_{L^2_{k,\nu}} = \left(\sum_{j=0}^k \| r^{-\frac{n}{p}-\nu+j} \nabla^j u \|^p_{L^p}\right)^{1/p}$

$$\|u\|_{C^{k,\alpha}_\nu} = \sum_{j=0}^k \|r^{-\nu+j} \nabla^j u\|_{C^0} + [r^{-\nu+k} \nabla^k u]_\alpha$$

Defn: For $\nu \in \mathbb{R}$ outside the discrete set of indicial roots of $d + d^*$

let $\mathcal{H}^k_\nu(B)$ be the finite dim'l space of closed & coclosed k-forms on B in C^∞_ν

Rmk (elliptic regularity) for $l \geq 0$, $p \geq 1$, $\alpha \in (0,1)$.

$$\mathcal{H}^k_\nu(B) = L^p_{l,\nu} \cap \ker(d + d^*) = C^{l,\alpha}_\nu \cap \ker(d + d^*)$$

Defn: For $\nu \in \mathbb{R}$ ------, let $\mathcal{W}^k_\nu$ be the L^2_ν-orthogonal complement of $\mathcal{H}^k_\nu \subset L^2_\nu(B, \Omega^k)$ $(\subset C^{0,\alpha}_\nu(B, \Omega^k))$.

§ Functions & 1-forms.

Let (B, ω_0, Ω_0) be an "irreducible" AC CY 3-fold, i.e. $\nexists$ parallel 1-form on B

Fact: $\tilde{\Sigma} \not\cong_{\text{isom}} (S^5, g_{std}) \Rightarrow (B, \omega_0, \Omega_0)$ irred.

Lem 5.6: For any $\nu < 0$, $\nexists$ non-zero harmonic fcns & 1-forms on B in C^∞_ν

pf. u, r harmonic fcn & 1-forms in C^∞_ν

Fact: $\nu < -2$, we can integration by part.

$\hookrightarrow \Delta : C^\infty_\nu \to C^\infty_{\nu-2}$.

$$0 = \langle \Delta u, u \rangle_{L^2} = \|\nabla u\|^2_{L^2} \quad \& \quad 0 = \langle \Delta r, r \rangle_{L^2} = \|\nabla r\|^2_{L^2}$$

rate < -4 rate < -2

$(-4) + (-2) < -6 + 1$

$$\Rightarrow \nabla u = 0 = \nabla r$$

$\Rightarrow$ u is constant, but u decay at $\infty \Rightarrow u = 0$.

∇r parallel 1-form (doesn't exist)

Prop 4.12/4.13 no harmonic function & 1-form homogeneous of order $\lambda \in [-2, 0)$

u harmonic of decay $O(r^\nu) \rightsquigarrow u = r^\lambda \phi + \text{l.o.t}$

λ indicial root eigen fcn on Σ

but $\Rightarrow \lambda < -2 \Rightarrow \nu < -2 \Rightarrow u = 0$

Similar for 1-forms #.

Lem 5.7. For $\nu < 0$, $\nexists$ fcns g & f & 1-forms r on B in C^∞_ν s.t.

$$\pi_1 dd^*(g\, \omega_0^2) = 0, \quad \pi_{1\oplus 6} dd^*(f \omega_0 + r^\# \lrcorner \operatorname{Re} \Omega_0) = 0$$

§ Closed & coclosed 2-forms & 3-forms.

Lem 5.11: Identify cohomology classes on Σ w/ their harmonic representative.
We have $H^2(\Sigma) = \mathrm{im}(H^2(B) \to H^2(\Sigma)) \oplus *_\Sigma \mathrm{im}(H^3(B) \to H^3(\Sigma))$.

Thm 5.12. Let B be an AC CY 3-fold asymp. to $C(\Sigma)$.
(i) $\forall \nu \in (-6,-2)$, $\mathcal{H}^2_\nu(B) \simeq H^2_c(B) \simeq L^2\mathcal{H}^2(B)$
(ii) $\forall \nu \in (-2,0)$, $\dim \mathcal{H}^2_\nu(B) = \dim H^2_c(B) + \dim \mathrm{im}(H^2(B) \to H^2(\Sigma))$
Moreover, $\forall$ harmonic 2-form τ on Σ with $\tau \in \mathrm{im}(H^2(B) \to H^2(\Sigma))$
$\exists\, \sigma \in \mathcal{H}^2_\nu(B)$ for every $\nu > -2$ s.t. $\forall \mu > 0$, $\sigma = \tau + O(r^{-2-\mu})$
In particular, $\mathcal{H}^2_\nu(B) \to H^2(B)$ is an isomorphism
(iii) $\forall$ sufficiently small $\delta > 0$, $\exists$ isomorphism $\mathcal{H}^3_{-3-\delta}(B) \simeq \mathrm{im}(H^3_c(B) \to H^3(B))$
$\simeq L^2\mathcal{H}^3(B)$
(iv) $\forall$ sufficiently small $\delta > 0$, every $\rho \in \mathcal{H}^3_{-3+\delta}(B)$ has an expansion of the form
$\rho = -\eta \wedge \tau_1 + \frac{1}{r} dr \wedge \tau_2 + O(r^{-3-\mu})$ for some small $\mu > 0$ & harmonic 2-forms
τ_1, τ_2 on Σ s.t. $*_\Sigma \tau_i = -\eta \wedge \tau_i$ represents a cohomology class in the image of $H^3(B) \to H^3(\Sigma)$.
Moreover, the map $\rho \mapsto ([\eta \wedge \tau_1], [\eta \wedge \tau_2])$ induces an isomorphism
$\mathcal{H}^3_{-3+\delta}(B) / \mathcal{H}^3_{-3-\delta}(B) \simeq \mathrm{im}(H^3(B) \to H^3(\Sigma)) \oplus \mathrm{im}(H^3(B) \to H^3(\Sigma))$.

Lem 5.13. Let (B, ω_0, Ω_0) be an irred. AC CY 3-fold. For $\nu \in \mathbb{R}$, denote by $\ker \Delta^2_\nu$ the space of harmonic 2-forms on B of class C^∞_ν.
(i) If $\nu < -2$, $\ker \Delta^2_\nu = \mathcal{H}^2_\nu(B)$, i.e. every harmonic 2-form of rate ν is closed & coclosed
(ii) If $\nu < 1$, every $\sigma \in \ker \Delta^2_\nu$ is coclosed.
In particular, $* d\sigma \in \mathcal{H}^3_{\nu-1}(B)$ for every $\sigma \in \ker \Delta^2_\nu$
(iii) The composition of $\sigma \mapsto [* d\sigma]$ & restriction $H^3(B) \to H^3(\Sigma)$ induces an isomorphism btw $\ker \Delta^2_{-2+\mu} / \mathcal{H}^2_{-2+\mu}(B)$ and $\mathrm{im}(H^3(B) \to H^3(\Sigma))$ $\forall$ sufficiently small $\mu > 0$.

Rmk 5.14 : Identify $H^2(\Sigma)$ w/ $\ker \Delta^2_\Sigma$

Lem 5.11 : $H^2(\Sigma) = \underbrace{\mathrm{im}\,(H^2(B) \to H^2(\Sigma))}_{=V_2} \oplus *_\Sigma \underbrace{\mathrm{im}\,(H^3(B) \to H^3(\Sigma))}_{=V_3}$

Choose L^2 onb $\tau_1, \dots, \tau_b \in \mathcal{H}^2(\Sigma)$, $b = b_2(\Sigma)$

with $\tau_i \in V_2$ for $i \leq k = \dim V_2$,

$\tau_i \in V_3$ $\quad i \geq k+1$

Thm 5.12 (ii)(iv) :

$\exists$ closed & co-closed 2-forms $\sigma_1, \dots, \sigma_k$ on B with $\sigma_i = \tau_i + O(r^{-2-\mu})$

3-forms $\rho_{k+1}, \dots, \rho_b$ on B $\quad \rho_i = \eta \wedge \tau_i + O(r^{-3-\mu})$

for some $\mu > 0$ $\quad$ η contact struc on Σ.

Thm 5.12 (i) (iii) :

Can make these choice unique & require $*\sigma_i$, ρ_i integrate to zero on the closed cycles, which are Poincaré dual to cohomology classes of closed & coclosed harmonic 2-form

Lem 5.13 (iii) :

For all $j = k+1, \dots, b$ $\exists!$ harmonic 2-form $\bar\sigma_j$ (modulo $\mathcal{H}^2_{-2+\mu}$ span$\{\sigma_1, \dots, \sigma_k\}$ & $L^2\mathcal{H}^2$)

s.t. $d\bar\sigma_j = *\rho_j$,

Asymp. of $\bar\sigma_j$: $d\bar\sigma_j = *\rho_j$ & $*\rho_j = -r^{-1} dr \wedge \tau_j + O(r^{-3-\mu})$

$\hookrightarrow$ $\bar\sigma_j = -(\log r)\,\tau_j + \sum_{i=k+1}^{b} \alpha^i_j \tau_i + O(r^{-2-\mu})$ for some $\alpha^i_j \in \mathbb{R}$.

and $\mu > 0$ sufficiently small.

Prop 5.16: Let (B, ω_0, Ω_0) be a irred AC CY 3-fold.
Fix $k \geq 1$, $\alpha \in (0,1)$, sufficiently small δ; $\nu > -3-\delta$ s.t. $\nu+1$ not an indicial root of $\Delta \curvearrowright \Omega^2$.
Then $\exists\, C > 0$ s.t. $\forall \psi \in C^{k,\alpha}_{\nu}$, $\exists\, \sigma$ w/ $d\sigma \in C^{k,\alpha}_{\nu}$ s.t. $\Delta\sigma = d^*\psi$
& $\|d\sigma\|_{C^{k,\alpha}_{\nu}} \leq C \|d^*\psi\|_{C^{k-1,\alpha}_{\nu-1}}$

Pf. It suffices to consider $k=1$ (elliptic regularity $\leadsto$ higher order)
Claim: the obstruction to solve $\Delta\sigma = d^*\psi$ with $\sigma \in C^{2,\alpha}_{\nu+1}$ lies in $\mathcal{H}^2_{-5-\nu}$, which are not closed.
($\Delta : C^{2,\alpha}_{\nu+1} \to C^{0,\alpha}_{\nu-1}$, obstruction: coker of this op. $= \ker \Delta^2_{\text{dual weight of } \nu-1 = \mu}$
$(\nu-1) + \mu + 5 < -1 \Leftrightarrow \mu < -5-\nu$
Take $\eta \in \mathcal{H}^2_{-5-\nu}$ $\leadsto \Delta\eta = 0$. consider $\langle d^*\psi, \eta\rangle_{L^2}$
if $d\eta = 0 \Rightarrow$ $= \langle \psi, d\eta\rangle_{L^2} = 0$)

① Lem 5.13(i): if $\nu > -3 \Rightarrow -5-\nu < -2$, $\ker \Delta^2_{-5-\nu} = \mathcal{H}^2_{-5-\nu}$
$\Rightarrow$ harmonic form of rate $-5-\nu$ are both closed & coclosed $\leadsto$ obstruction $=0$
$\Rightarrow \exists!$ 2-form $\sigma \in C^{2,\alpha}_{\nu+1}$, $L^2_{\nu+1}$-orthogonal to $\mathcal{H}^2_{\nu+1}$ st. $\Delta\sigma = d^*\psi$.

② If $\nu \in (-3-\delta, -3)$ & $H^2(B) \to H^2(\Sigma)$ is surjective
$\hookrightarrow -5-\nu \in (-2, -2+\delta)$
$H^2(\Sigma) = \mathrm{im}(H^2(B) \to H^2(\Sigma)) \oplus *_\Sigma \underbrace{\mathrm{im}(H^3(B) \to H^3(\Sigma))}_{=0}$
Lem 5.13(iii)
$\Rightarrow \ker \Delta^2_{-5-\nu} / \mathcal{H}^2_{-5-\nu} \simeq \mathrm{im}(H^3(B) \to H^3(\Sigma)) = 0$
$\Rightarrow$ harmonic 2-forms of rate $-5-\nu$ are both closed & coclosed.

③ If $\nu \in (-3-\delta, -3)$ & $H^2(B) \to H^2(\Sigma)$ is not surjective.
↗ Note that $\{\bar{\sigma}_{k+1}, \ldots, \bar{\sigma}_b\}$ is a basis of the sp of obstructions

$\rightsquigarrow$ might have obstructions to solve $\Delta\sigma = d^*\psi$ with $\sigma \in C^{2,\alpha}_{\nu+1}$

But for μ small, one can solve $\Delta\sigma = d^*\psi$ for $\sigma \in C^{2,\alpha}_{-2+\mu}$ uniquely up to $\mathcal{I}$ harmonic form

$\boxed{\text{WTS}}$: Can choose σ s.t. $d\sigma \in C^{1,\alpha}_{\nu}$

Consider harmonic 2-forms $\bar{\sigma}_{k+1}, \ldots, \bar{\sigma}_b$ (closed) in Rmk 5.14

Fix χ cutoff on B s.t. $\chi \equiv 1$ on $B \setminus K$.

$\forall\ j = k+1, \ldots, b$, set $\sigma_j' = \chi \cdot \tau_j$, τ_j harmonic on $\Sigma \rightsquigarrow$ on $C(\Sigma)$

Since τ_i closed & coclosed $\Rightarrow d\sigma_j' = d\chi \wedge \tau_i$ cptly supported
& $d^*\sigma_j' \in C^\infty_\nu$

Note that $\bar{\sigma}_h \in C^\infty_{-2+\mu}$ for any $\mu > 0$

Since $\nu < -3$, we can fix $\mu > 0$ s.t. $\nu + (-2+\mu) < -5$.

$$\langle \Delta\sigma_j', \bar{\sigma}_h \rangle_{L^2} \overset{\substack{\text{IPP from Lem B5} \\ \downarrow}}{=} \langle d^*\sigma_j', \cancelto{0}{d^*\bar{\sigma}_h} \rangle_{L^2} + \langle d\sigma_j', \overbrace{d\bar{\sigma}_h}^{= *\rho_h} \rangle_{L^2}$$

$$= \langle d\sigma_j', *\rho_h \rangle_{L^2}$$

$$d\rho_h = 0 \curvearrowright\ = -\lim_{R\to\infty} \int_{r \le R} d(\sigma_j' \wedge \rho_h) = -\lim_{R\to\infty} \int_{r \ge R} \tau_j \wedge (\eta \wedge \tau_h + O(r^{-3-\mu}))$$

$$= -\int_\Sigma \tau_j \wedge \eta \wedge \tau_h = \delta_{jh}$$

Given $\psi \in C^{1,\alpha}_\nu$, set $a_i = \langle d^*\psi, \bar{\sigma}_i \rangle_{L^2}$

Note:
- $|d^*\psi| \le \|d^*\psi\|_{C^{0,\alpha}_{\nu-1}} \cdot r^{\nu-1}$
- $|\bar{\sigma}_i| \le C r^{-2} \log r$
- $r^{\nu-3} \log r$ integrable when $\nu < -3$

$\Big\} \Rightarrow |a_i| \le C \cdot \|d^*\psi\|_{C^{0,\alpha}_{\nu-1}}$

Consider $d^*\varphi - \sum_{i=k+1}^{b} a_i \Delta\sigma_i' \in C^{0,\alpha}_{\nu-1}$

$\Rightarrow \langle d^*\varphi - \sum_{i=k+1}^{b} a_i \Delta\sigma_i', \bar{\sigma}_h \rangle_{L^2} = 0 \quad \forall\, h = k+1, \dots, b$

$\Rightarrow \exists!$ 2-form $\sigma' \in C^{2,\alpha}_{\nu+1}$ which is $L^2_{\nu+1}$ o.n. to harmonic forms

& $\Delta\sigma' = d^*\varphi - \sum_{i=k+1}^{b} a_i \Delta\sigma_i'$

$$\|\sigma'\|_{C^{2,\alpha}_{\nu+1}} \leq C\Big(\|d^*\varphi\|_{C^{0,\alpha}_{\nu-1}} + \sum_{i=k+1}^{b} |a_i| \overbrace{\|\Delta\sigma_j'\|_{C^{0,\alpha}_{\nu-1}}}^{\text{unif bdd by construction}}\Big)$$

$$\leq C\, \|d^*\varphi\|_{C^{0,\alpha}_{\nu-1}}$$

Finally, set $\sigma = \sigma' + \sum_{i=k+1}^{b} a_i \sigma_i'$ #

§ Normal forms for exact 4-forms

$\llcorner$ interplay btw mapping properties of operator $d + d^*$

& type-decomposition of differential forms on AC CY-3 fold

also be used to understand the image of the linearization of Apostolov-Salamon eqn.

Lem 5.17.: For $\nu \in (-6,-1)$, $k \geq 0$, $\alpha \in (0,1)$, $\exists$ a constant $C>0$ st for any 3-form $\rho \in C^{k,\alpha}_{\nu}$, $\exists !$ $\rho_0 \in C^{k,\alpha}_{\nu} \cap \Omega^3_{12}$ and $f, g, \gamma \in C^{k+1,\alpha}_{\nu+1}$

with $\|(f,g,\gamma)\|_{C^{k+1,\alpha}_{\nu}} + \|\rho_0\|_{C^{k,\alpha}_{\nu}} \leq C\|\rho\|_{C^{k,\alpha}_{\nu}}$.

& $\rho = d(f\omega_0 + \gamma^{\#} \lrcorner \operatorname{Re}\Omega_0) + d^*(\frac{1}{2} g\, \omega_0^2) + \rho_0$

Recall that : $\mathcal{W}^3_{\nu} = L^2$-o.n. complement of $\mathcal{H}^3_{\nu}$ in $C^{0,\alpha}_{\nu}$

Prop 5.18: Let (B, ω_0, Ω_0) be an irreducible AC Calabi-Yau 3 fold, asymp. to $C(\Sigma)$. Fix $k \geq 1$, $\alpha \in (0,1)$, $\delta > 0$ (as in Prop 5.16), $\nu \in (-3-\delta, -1)$ away from discrete set of indicial roots.

Then every exact 4-form $\sigma = d\rho'$ with $\rho' \in C^{k,\alpha}_{\nu}$ can be written as

$\sigma = d * d(f + \gamma^{\#} \lrcorner \operatorname{Re}\Omega) + d\rho_0$

where $f, \gamma \in C^{k+1,\alpha}_{\nu+1}$, $\rho_0 \in C^{k,\alpha}_{\nu} \cap \mathcal{W}^3_{\nu} \cap \Omega^3_{12}$ with $d^*\rho_0 = 0$

and $\|(f,\gamma)\|_{C^{k+1,\alpha}_{\nu+1}} + \|\rho_0\|_{C^{k,\alpha}_{\nu}} \leq C\|\rho'\|_{C^{k,\alpha}_{\nu}}$

for some constant $C>0$ indep of ρ'.

Moreover $f = 0 = \gamma$ if $\sigma \in \Omega^4_8$.

pf. Recall: $\Lambda^3\mathbb{R}^6 = \Lambda^3_6 \oplus \Lambda^3_{1\oplus1} \oplus \Lambda^3_{12}$, $\Lambda^3_6 = \{X^{\flat} \wedge \omega \mid X \in \mathbb{R}^6\}$

$\Lambda^3_{1\oplus1} = \mathbb{R}\operatorname{Re}\Omega \oplus \mathbb{R}\operatorname{Im}\Omega$

Λ^3_{12} = primitive forms of type $(1,2)+(2,1)$

$\Lambda^2\mathbb{R}^6 = \Lambda^2_1 \oplus \Lambda^2_6 \oplus \Lambda^2_8$, $\Lambda^2_1 = \mathbb{R}\omega$, $\Lambda^2_6 = \{X \lrcorner \operatorname{Re}\Omega \mid X \in \mathbb{R}^6\}$

Λ^2_8 = primitive forms of type $(1,1)$

Pf. ① If we can write $\sigma = d\rho$ for some 3-form $\rho \in C^{k,\alpha}_\nu$ with $d^*\rho = 0$.

$\Rightarrow \rho$ is uniquely defined up to $\pm$ closed & coclosed 3-form in $C^{k,\alpha}_\nu$

$\rho = \underbrace{\gamma \wedge \omega}_{\Omega^3_6} + \underbrace{\lambda \,\mathrm{Re}\,\Omega + \mu \,\mathrm{Im}\,\Omega}_{\Omega^3_{1\oplus 1}} + \underbrace{\rho_0}_{\Omega^3_{12}}$ $\Rightarrow$ $\Delta\gamma = 0$, $\Delta\lambda = 0 = \Delta\mu$

ρ harmonic

but no harmonic fcn & 1-form in C^∞_ν for $\nu < 0$

② $\sigma = d\rho'$ for some $\rho' \in C^{k,\alpha}_\nu$

Prop 5.16: $\exists$ 2-form α s.t. $\Delta\alpha = -d^*\rho'$ and $d\alpha \in C^{k,\alpha}_\nu$

We know: $\begin{cases} \alpha \in C^{k+1,\alpha}_{\nu+1} & \text{if } \nu \geq -3 \text{ or } H^2(B) \to H^2(\Sigma) \text{ surjective.} \\ \alpha \in C^{k+1,\alpha}_{-2+\mu} & \text{for any } \mu > 0 \text{ small, otherwise} \end{cases}$

Since $\nu < 0$, $\left.\begin{aligned} d^*\alpha &\in C^{k,\alpha}_\nu \text{ ---} \\ &\in C^{k,\alpha}_{-3+\mu} \text{ ---} \end{aligned}\right\} \Rightarrow d^*\alpha$ decay harmonic 1-form

Lem 5.6 $\Rightarrow d^*\alpha = 0$, Hence, $\rho = \underbrace{\rho' + d\alpha}_{\in C^{k,\alpha}_\nu}$ is coclosed & $d\rho = \sigma$

$\leadsto$ give the form needed in ①

③ Lem 5.17: $*\rho = -d(f\omega_0 + \gamma^\# \mathrm{Re}\,\Omega_0) + \frac{1}{2} d^*(g\omega_0^2) + *\rho_0$

with $\rho_0 \in \Omega^3_{12} \cap C^{k,\alpha}_\nu$ & $f, g, \gamma \in C^{k+1,\alpha}_{\nu+1}$

ρ is uniquely defined up to $\pm$ closed & coclosed 3-form in Ω^3_{12}

$\Rightarrow$ can make ρ unique by asking $\rho_0 \in W^3_\nu$

In particular,

$0 = d*\rho = \frac{1}{2} dd^*(g\omega_0^2) + d*\rho_0$

$\Rightarrow 0 = \pi_1\left(\frac{1}{2} dd^*(g\omega_0^2) + d*\rho_0\right) = \pi_1 dd^*\left(\frac{1}{2} g\omega_0^2\right)$

$d(*\rho_0) \wedge \omega_0 = d(\underbrace{*\rho_0 \wedge \omega_0}_{=0 \text{ by Lem 2.8}})$

Lem 2.19 $\Rightarrow g$ harmonic then $= 0$ since $\nu + 1 < 0$

Hence $d^*\rho_0 = d^*\rho = 0$.

④ Assume moreover $\sigma = d*d(f\omega_0 + \gamma^\# \lrcorner \operatorname{Re}\Omega_0) + d\rho_0 \in \Omega^4_8$

Since $d^*\rho_0 = 0 \Rightarrow d\rho_0 \in \Omega^4_8$ by Rmk 2.18

$\Rightarrow d*d(f\omega_0 + \gamma^\# \lrcorner \operatorname{Re}\Omega_0) \in \Omega^4_8$

Lem 2.19: $\Delta f = 0$ & $dd^*\gamma + \frac{2}{3}d^*d\gamma = 0$

Lem 5.7: $f = 0$, $\gamma = 0$ for $\nu < 0$ #

Cor 5.19: Same condition as in Prop 5.19, we can write

Prop 5.15 $\to$ $\sigma = d*d(f\omega_0) + d(*d(\gamma^\# \lrcorner \operatorname{Re}\Omega_0) - d(\gamma^\# \lrcorner \operatorname{Im}\Omega_0)) + d\rho_0$

Lem 2.16 $\to$ Moreover $*d(\gamma^\# \lrcorner \operatorname{Re}\Omega_0) - d(\gamma^\# \lrcorner \operatorname{Im}\Omega_0) \in \Omega^3_{12}$