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Familles de métriques hermitiennes canoniques

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Résumé

Dans cette thèse, nous nous intéressons aux métriques singulières canoniques et spéciales sur une famille de variétés compactes complexes possiblement non-kählériennes.

Le premier résultat concerne l'existence de métriques de Gauduchon "singulières". Sous certaines hypothèses sur les singularités, nous prouvons qu'une telle variété singulière compacte admet une métrique de Gauduchon bornée dans la classe conforme d'une métrique hermitienne lisse fixée. Ceci généralise le théorème de Gauduchon dans le cas singulier. D'autre part, dans chaque classe conforme, nous montrons également l'unicité des métriques de Gauduchon bornées à une constante multiplicative près. De plus, nous obtenons une propriété d'extension de la $(n-1)$ème puissance d'une métrique de Gauduchon bornée comme courant plurifermé positif sur une variété singulière.

Dans la deuxième partie, nous cherchons à établir des estimations uniformes pour les solutions d'équations de Monge–Ampère complexes pour une famille de variétés hermitiennes. Ensuite, nous montrons une estimation uniforme L^1 des fonctions quasi-plurisousharmoniques sup-normalisées quand la famille est un lissage d'une variété hermitienne qui n'a que des singularités isolées. La preuve s'appuie sur la théorie du pluripotentiel et un contrôle uniforme des facteurs conformes de Gauduchon. Combinée à l'estimation uniforme L^p des densités canoniques, nous obtenons un contrôle uniforme des solutions des familles d'équations de Monge–Ampère complexes dans de tels cas.

Mots clés

Équations de Monge–Ampère complexes, Familles d'espaces complexes, Métriques de Gauduchon, Variétés de Calabi–Yau, Métriques de Chern–Ricci plates.

Abstract

In this thesis, we are interested in singular canonical and special metrics on a family of compact complex possibly non-Kähler varieties.

The first result concerns the existence of "singular" Gauduchon metrics. Under certain assumptions on the singularities, we prove that such a compact singular variety admits a bounded Gauduchon metric in every conformal class of a smooth hermitian metric. This generalizes Gauduchon's theorem to a singular setup. On the other hand, in each conformal class, we also show the uniqueness of bounded Gauduchon metrics up to a positive multiple in each conformal class. Furthermore, we obtain an extension property of the $(n-1)$-th power of a bounded Gauduchon metric as a positive pluriclosed current on a singular variety.

In the second part, we aim at establishing a uniform L^∞-estimate of solutions to complex Monge–Ampère equations for a family of hermitian varieties. Then we show a uniform L^1-estimate of sup-normalized quasi-plurisubharmonic functions when the family is a smoothing of a hermitian variety that has only isolated singularities. The proof consists of pluripotential theory and uniform control of Gauduchon factors. Furthermore, combining with a uniform L^p-estimate of the canonical densities, we obtain uniform boundedness of the solutions of families of complex Monge–Ampère equations in such cases.

Keywords

Complex Monge–Ampère equations, Families of complex spaces, Gauduchon metrics, Calabi–Yau varieties, Chern–Ricci flat metrics.

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自強不息

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《易經．乾卦．象》

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Chapitre 1

Introduction (en français)

1.1 Motivation

Dans cette thèse, nous étudions des familles de métriques hermitiennes spéciales. La compréhension de l'existence de métriques canoniques et de leur dégénérescence est un sujet de recherche très actif ; il a de nombreuses interactions avec plusieurs branches différentes des mathématiques et de la physique (cf. [KS01, GHJ03, YN10]).

1.1.1 Métriques kählériennes canoniques

En analyse complexe, le concept de métrique canonique apparaît dans le théorème d'uniformisation de Poincaré, qui stipule que toute surface de Riemann simplement connexe est équivalente de manière conforme à la sphère de Riemann, au plan complexe ou au disque unitaire dans le plan complexe. Chaque classe de métrique conforme contient une métrique canonique (métrique à courbure constante) sur les surfaces de Riemann compactes. D'autre part, suivant le théorème de Gauss–Bonnet

$$\int_\Sigma K_g \mathrm{d}A_g = 2\pi\chi(\Sigma),$$

le signe de la courbure gaussienne K_g d'une métrique canonique g dépend de la topologie (la caractéristique d'Euler) de la surface Σ. Par conséquent, sur les surfaces de Riemann compactes, elles relient des quantités topologiques et géométriques qui semblent sans rapport à première vue.

Il existe plusieurs généralisations possibles des métriques à courbure constante en dimension supérieure. Les métriques de Kähler–Einstein sont des analogues prototypiques de ces métriques canoniques. Sur une variété kählérienne compacte X, une métrique de Kähler–Einstein est une métrique kählérienne ω dont la forme de Ricci est proportionnelle à la forme de Kähler, à savoir

$$\mathrm{Ric}(\omega) = \lambda\omega$$

pour un certain $\lambda \in \mathbb{R}$. Quite à dilater ω, on peut toujours supposer que $\lambda \in \{-1, 0, 1\}$.

L'existence d'une métrique de Kähler–Einstein impose une forte contrainte cohomologique : la première classe de Chern a un signe, $c_1(X) < 0$ (fibré canonique est ample), ou $c_1(X) = 0$ (Calabi–Yau), ou encore $c_1(X) > 0$ (Fano). Un fibré en droites holomorphe L sur X peut être représenté à isomorphisme près par une classe dans $H^1(X, \mathcal{O}_X^*)$, et la première classe de Chern est donnée par l'homomorphisme de connexion $c_1 : H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z})$ induit par la suite exacte courte $0 \to \mathbb{Z} \to \mathcal{O}_X \xrightarrow{e^{2pii\bullet}} \mathcal{O}_X^* \to 1$ de faisceaux. La première classe de Chern $c_1(X) \in H^{1,1}(X, \mathbb{R}) \cap H^2(X, \mathbb{Z})$ d'une variété complexe X est la première classe de Chern du fibré anticanonique $K_X^{-1} = \bigwedge^n T_X$, où n est la dimension complexe de X.

Conjecture de Calabi et Conjecture de Yau–Tian–Donaldson

Pour une forme kählérienne fixe ω, sa forme de Ricci $\mathrm{Ric}(\omega)$, définie localement comme $\mathrm{i}\partial\bar{\partial}\log(\omega^n)$, est toujours une forme fermée $(1,1)$ sur X, dont la classe de cohomologie de de Rham est indépendante de ω: c'est un représentant de la classe $2\pi c_1(X)$. Dans les années 1950, Calabi [Cal57] a posé la question inverse :

Conjecture (Conjecture de Calabi)**.** Soit (X, ω) une variété kählérienne compacte. Si l'on considère une forme lisse $(1,1)$ $\Psi \in 2\pi c_1(X)$, existe-t-il une métrique kählérienne ω' cohomologuée à ω telle que

$$\mathrm{Ric}(\omega') = \Psi?$$

Calabi a montré que ω' est unique s'il existe, et a relié l'existence à la résolution d'une EDP non linéaire. La conjecture de Calabi a été prouvée par Yau [Yau78] en 1978. En utilisant la méthode de continuité et certaines estimations a priori, Yau a complètement résolu l'EDP non linéaire, l'équation de Monge–Ampère complexe, comme suit

$$(\omega + \mathrm{i}\partial\bar{\partial}\varphi)^n = f\omega^n \tag{MA}$$

où φ est l'inconnue, f est une fonction lisse strictement positive dépendant de ω et Ψ, et f satisfait aussi naturellement une condition normalisée $\int_X (f-1)\omega^n = 0$. De même, la recherche d'une métrique de Kähler–Einstein se résume à la résolution d'une équation de Monge–Ampères complexe dont le membre de droite peut dépendre de l'inconnue :

$$(\omega + \mathrm{i}\partial\bar{\partial}\varphi)^n = e^{-\lambda\varphi} f\omega^n.$$

Sur une variété avec fibré canonique ample, Aubin et Yau [Aub78, Yau78] ont indépendamment montré l'existence d'une unique métrique de Kähler–Einstein dans $c_1(K_X)$. Dans le cas de Calabi–Yau, par le théorème de Yau, il existe une métrique kählérienne Ricci plate unique dans chaque classe kählérienne. Cependant, si X est Fano, le problème de Kähler–Einstein est plus subtil. Une obstruction à l'existence de métriques de Kähler–Einstein à courbure positive a été découverte par Matsushima [Mat57] : Le groupe d'automorphisme $\mathrm{Aut}(X)$ d'une variété de Fano Kähler–Einstein X est réductif. Une autre obstruction, liée aux champs de vecteurs holomorphes, a été trouvée par Futaki [Fut83]. En dimension complexe deux, le problème de Kähler–Einstein a été complètement résolu par Tian et Yau [Tia87, TY87]. Introduite par Yau et Tian [Tia97] et ensuite affinée par Donaldson [Don02]:

Conjecture (Conjecture de Yau–Tian–Donaldson)**.** Une variété de Fano $(X, -K_X)$ admet une métrique de Kähler–Einstein si et seulement s'il satisfait à une propriété algébrique appelée K-stabilité.

En 2015, Chen, Donaldson et Sun [CDS15] ont résolu la conjecture de Yau–Tian–Donaldson sur les variétés de Fano. Ce résultat fait le pont entre un objet de géométrie différentielle et une notion purement algébro-géométrique. Plusieurs preuves sont apparues par la suite : [DS16, CSW18, BBJ21, Zha21, Li22].

Uniformisation des variétés de Kähler–Einstein

L'inégalité de Chen–Ogiue donne une caractérisation des variétés de Kähler–Einstein, qui ont des courbures constantes en termes de nombres de Chern. Cela donne également un critère d'uniformisation dans les dimensions supérieures:

Théorème ([CO75])**.** *Si (X, ω) est une variété compacte de Kähler–Einstein de dimension complexe $n \geq 2$, alors*

$$\left(2(n+1)c_2(X) - nc_1(X)^2\right)[\omega]^{n-2} \geq 0.$$

De plus, l'égalité tient si et seulement si ω a une courbure bissectionnelle holomorphe constante ; et, l'espace de recouvrement universel $\widetilde{X}$ de X est isomorphe à $\mathbb{CP}^n$, $\mathbb{C}^n$, ou $\mathbb{B}^n$ (selon le signe de la courbure).

Il existe de nombreuses applications géométriques de la métrique de Kähler–Einstein. Nous renvoyons le lecteur à [Tia00] et aux références qui y figurent.

1.1.2 Résolution des équations de Monge–Ampère complexes sur les variétés kählériennes compactes

Comme nous l'avons mentionné dans la section précédente, la résolution d'équations de Monge–Ampère complexes est importante pour comprendre l'existence des métriques de Kähler–Einstein. Dans cette section, nous passons en revue deux stratégies pour résoudre les équations de Monge–Ampère complexes (MA).

Méthode de continuité

Dans [Yau78], Yau a résolu (MA) en utilisant une technique d'EDP, appelée la méthode de continuité. La méthode de continuité consiste à résoudre d'abord une équation facile, puis à déformer la solution pour obtenir l'équation souhaitée. Précisément, pour $t \in [0,1]$, on considère la famille d'équations suivante

$$(\omega + i\partial\bar{\partial}\varphi_t)^n = c_t e^{t\log f}\omega^n, \quad \omega + i\partial\bar{\partial}\varphi_t > 0, \quad \text{and } \sup_X \varphi_t = 0 \qquad (\mathrm{MA}_t)$$

où c_t est une constante de normalisation de sorte que $c_t \int_X e^{t\log f}\omega^n = \int_X \omega^n$. Désignons par

$$S := \{t \in [0,1] \mid (\mathrm{MA}_t) \text{ admet une solution lisse}\}.$$

La preuve de la méthode de continuité consiste à vérifier trois parties : S est non vide ; S est ouvert dans $[0,1]$; S est fermé dans $[0,1]$. Puisque $\varphi_0 \equiv 0$ est une solution de (MA_t) à $t = 0$, S est non vide. L'ouverture peut être obtenue par le théorème des fonctions implicites des espaces de Banach et certains arguments de perturbation. Le principal challenge est de prouver la fermeture. Pour montrer que S est fermé, Yau a établi certaines estimations a priori aux équations (MA_t). La partie la plus cruciale et la plus difficile est une estimation uniforme de L^∞ qui a été surmontée par Yau, en utilisant l'itération de Moser. Soulignons que si l'on applique soigneusement l'approche de Yau, l'estimation de L^∞ ne nécessite qu'un contrôle de L^p sur la densité pour un certain $p > n$.

Preuve pluripotentielle

En 1998, Kołodziej [Koł98] a produit une estimation de L^∞ pour les fonctions de densité f qui sont simplement dans $L^{1+\varepsilon}$ pour un certain $\varepsilon > 0$. Au lieu d'utiliser l'itération de Moser, Kołodziej a combiné l'approche classique de De Giorgi et la théorie pluripotentielle. Dans sa méthode, Kołodziej a considéré une fonction définie par la capacité de Monge–Ampère de l'ensemble de niveau

$$g(s) := \mathrm{Cap}_\omega(\{\varphi < -s\})$$

où la capacité d'un ensemble E est définie par

$$\mathrm{Cap}_\omega(E) := \sup\left\{\int_E (\omega + i\partial\bar{\partial}u)^n \,\middle|\, u \in \mathrm{PSH}(X,\omega),\ -1 \le u \le 0\right\}.$$

Ici $\mathrm{PSH}(X,\omega)$ est l'espace de toutes les fonctions ω-plurisousharmoniques (voir la définition précise dans la section 1.2.2). Par l'approche de De Giorgi, $g(s)$ disparaît pour $s \ge s_\infty$; ceci implique que $\varphi \ge -s_\infty$. De plus, Kołodziej a également montré l'existence d'un potentiel continu unique $\varphi \in \mathrm{PSH}(X,\omega)$ qui résout l'équation de Monge–Ampère complexe avec densités dans L^p. C'est extrêmement utile pour les applications aux milieux singuliers. Eyssidieux, Guedj et Zeriahi [EGZ09] ont généralisé l'estimation pour permettre à la forme d'arrière-plan ω d'être semi-positive et grande ; les auteurs ont également prouvé le théorème de Calabi–Yau singulier sur les variétés "faiblement" singulières (voir la Section 1.2.1 pour les détails).

Nous allons maintenant mentionner brièvement une approche variationnelle telle que développée par [BBGZ13], même si elle n'est pas directement liée aux parties principales de cette thèse. La preuve de Kołodziej se réduit finalement par régularisation au cas lisse traité par Yau, mais la méthode variationnelle peut être rendue indépendante de la preuve EDP de Yau. Cependant, elle donne des solutions faibles qui n'ont pas besoin d'être lisses (bien que, en fin de compte, le théorème d'unicité montre que la solution de Yau coïncide avec la solution faible ; elle est donc lisse tant que la densité est lisse et positive). L'idée clé de l'approche variationnelle provient de l'analyse de la convexité de certaines fonctionnelles. Certaines de leurs équations d'Euler–Lagrange sont des équations de Monge–Ampère complexes. Une fonctionnelle importante est appelée la fonctionnelle d'énergie (ou fonctionnelle d'Aubin–Mabuchi), définie comme suit

$$\mathbf{E}(u) = \frac{1}{(n+1)V} \sum_{j=0}^{n} \int_X u(\omega + \mathrm{i}\partial\bar{\partial}u)^j \wedge \omega^{n-j},$$

où $V = \int_X \omega^n$. Cette fonctionnelle et ses variantes ont permis de réaliser des progrès importants dans l'étude d'une géométrie métrique de l'espace des potentiels de Kähler et dans la construction d'estimations a priori uniformes (cf. [Dar15, DR17, DG18, BBE+19, BBJ21]).

1.1.3 Cas hermitiens

Étant donné que les variétés non-kählériennes apparaissent fréquemment dans diverses constructions, l'étude des équations de Monge–Ampère complexes sur les variétés hermitiennes a suscité un intérêt considérable au cours de la dernière décennie. Sur les variétés hermitiennes compactes, l'étude des équations de Monge–Ampère complexes a été entreprise pour la première fois par Cherrier [Che87] et Hanani [Han96]. En 2010, Tosatti et Weinkove [TW10] ont résolu une version importante et appropriée du théorème de Yau. Ils ont prouvé que, sur une variété hermitiennes compacte (X, ω), pour tout $0 < f \in \mathcal{C}^\infty(X)$, il existe une paire unique (φ, c) avec φ une fonction lisse et $c > 0$ une constante telle que

$$(\omega + \mathrm{i}\partial\bar{\partial}\varphi)^n = cf\omega^n, \quad \omega + \mathrm{i}\partial\bar{\partial}\varphi > 0, \quad \sup_X \varphi = 0.$$

En conséquence, ils ont également obtenu que, sur une variété de Calabi–Yau non-kählérienne, il existe toujours une métrique de Chern–Ricci plate en résolvant une équation de Monge–Ampère complexe de densité canonique. Comme dans la preuve originale de Yau, la méthode de [TW10] consiste à établir des estimations a priori le long d'un chemin de continuité, et l'estimation la plus délicate s'avère à nouveau être l'estimation a priori de L^∞. De plus, la non-fermeture de la forme de référence introduit plusieurs nouvelles difficultés : il y a beaucoup de termes supplémentaires à manipuler lors de l'utilisation du théorème de Stokes et il devient non trivial d'obtenir des limites uniformes sur le volume de Monge–Ampère. Le contrôle de la constante de normalisation c devient une question délicate.

La théorie du pluripotentiel a été partiellement étendue par Dinew, Kołodziej, et Nguyen [DK12, KN15, Din16, KN19] pour une densité f simplement dans L^p. Récemment, Guedj et

Lu [GL21] ont étendu le résultat d'existence de Tosatti et Weinkove aux variétés hermitiennes faiblement singulières. Ceci implique un théorème singulier de Calabi–Yau sur les variétés hermitiennes qui a généralisé le travail fondamental de Eyssidieux–Guedj–Zeriahi [EGZ09]. Il est intéressant de se demander comment ces métriques hermitiennes aident à mieux comprendre les variétés de Calabi-Yau non-kählériennes.

1.1.4 Pourquoi des singularités ?

Pour classifier les variétés complexes compactes, les singularités apparaissent de manière omniprésente dans plusieurs contextes, notamment dans la théorie des modules et le programme des modèles minimaux (MMP en abrégé, ou programme de Mori).

La MMP vise à classer les variétés algébriques jusqu'à l'équivalence birationnelle (ou biméromorphique). Le but est de trouver dans chaque classe birationnelle une variété qui est plus simple dans un certain sens, puis d'étudier ces "bons" modèles en détail. Un bon modèle est une variété qui satisfait l'une des hypothèses suivantes : K_X est nef (par exemple, dans les cas algébriques, $K_X.C \geq 0$ pour toute courbe irréductible C), ou K_X est semi-ample (ce que peut être équivalent à la première hypothèse cf. conjecture d'abondance), ou c'est un espace de fibres de Fano (ou de Mori). En 1975, Ueno [Uen75] a trouvé une classe birationnelle de variétés kählériennes compactes en trois dimensions qui n'admet pas de modèle minimal lisse, alors qu'elle admet un modèle minimal avec des "singularités faibles" (cf. Section 1.2.1).

En théorie des modules, il est nécessaire de traiter les variétés singulières lors de la compactification de l'espace des modules des variétés lisses. Par exemple, en dimension un, le domaine fondamental de l'espace de modules des courbes elliptiques est non compact et les courbes nodales se trouvent sur sa limite. Dans l'étude des modules de Calabi–Yau, une transition géométrique importante, appelée transition conifold, a été introduite par Clemens [Cle83] et Friedman [Fri86]. Reid [Rei87] a conjecturé que les variétés kählériennes de Calabi–Yau en dimension trois peuvent être connectées par des transitions de conifold. D'autre part, par l'intermédiaire de la transition conifold, de grandes classes de variétés non-kählériennes de Calabi-Yau pourraient apparaître même à partir d'une variété projective Calabi-Yau en dimension trois. Il est donc intéressant d'étudier les métriques canoniques sur les variétés non-kählériennes et leur dégénérescence en familles. Une question très générale est abordée comme suit : Soit $\pi : \mathcal{X} \to \mathcal{Y}$ une application holomorphe propre, surjective avec des fibres connectées, entre deux variétés. On voudrait relier la géométrie de $\mathcal{X}$ à celle de $\mathcal{Y}$ et aux fibres X_y de π.

Le cadre suivant a été considéré par plusieurs auteurs (cf. [Wan97, RZ11, RZ13, Tos15]). Soit $\pi : \mathcal{X} \to \mathbb{D}$ une famille projective plate de variétés en dimension n sur le disque unitaire $\mathbb{D} \subset \mathbb{C}$, avec $\mathcal{X}$ normale, et le diviseur canonique relatif $K_{\mathcal{X}/\mathbb{D}}$ trivial. Il s'ensuit que $X_t = \pi^{-1}(t)$ est une variété de Calabi–Yau pour $t \neq 0$ puisque son fibré canonique est trivial. Fixons un fibré en droites relativement ample $\mathcal{L} \to \mathcal{X}$ et fixons $L_t = \mathcal{L}_{|X_t}$. D'après le théorème de Yau, pour chaque $t \neq 0$, on peut trouver une unique métrique kählérienne Ricci plate ω_t appartenant à $c_1(L_t)$. Si les singularités de la fibre centrale X_0 sont "douces" (voir Section 1.2.1), un puissant théorème dû à Eyssidieux–Guedj–Zeriahi [EGZ09] garantit l'existence d'une unique métrique Ricci plate singulière ω_0 dans $c_1(L_0)$. Il est naturel de s'interroger sur le comportement de ω_t lorsque $t \to 0$.

C'est un problème ouvert fondamental pour mieux comprendre la géométrie de l'espace métrique incomplet $(X_0^{\text{reg}}, \omega_0)$, le comportement asymptotique de la métrique Ricci plate singulière près de X_0^{sing}, et le comportement de (X_t, ω_t) dans des paramètres plus singuliers. Dans le cadre de Calabi–Yau ci-dessus, Rong et Zhang [RZ13] ont montré que les espaces métriques (X_t, ω_t) convergent au sens de Gromov–Hausdorff vers un espace métrique compact (Z, d_Z) qui est isométrique à la complétion de l'espace métrique $(X_0^{\text{reg}}, \omega_0)$. De plus, suivant les travaux de Donaldson et Sun [DS14], Z est homéomorphe à X_0. D'autre part, une description très précise du comportement asymptotique a été obtenue par Hein et Sun dans [HS17] lorsque les singularités sont isolées et de plus isomorphes à un cône de Calabi–Yau lissable fortement régulier. Il est hautement souhaitable d'étendre ces descriptions à des situations plus générales (cf. [CS22]).

1.2 Préliminaires

Dans cette partie, nous passons en revue quelques notions que nous rencontrerons tout au long de la thèse.

1.2.1 Un rapide aperçu sur les espaces complexes singuliers

Résolution des singularités

Les espaces analytiques complexes sont des généralisations naturelles des variétés complexes. Une variété complexe lisse est construite en assemblant des parties ouvertes isomorphes à des ensembles ouverts dans un espace de coordonnées complexe. Un espace analytique complexe est obtenu en collant ensemble des parties ouvertes isomorphes à des sous-ensembles analytiques (c-à-d des zéros de fonctions holomorphes dans $\mathbb{C}^N$). Par variété, on entend un espace complexe irréductible, réduit, de dimension pure. Nous désignerons par X^{reg} la variété complexe lisse des points réguliers de X, et l'ensemble $X^{\text{sing}} := X \setminus X^{\text{reg}}$ des points singuliers qui est un sous-ensemble analytique de X de codimension complexe ≥ 1. Grâce à Hironaka [Hir64], X admet une résolution de singularités, à savoir qu'il existe une variété complexe lisse Y et une application biméromorphe surjective propre $\pi : Y \to X$. De plus, on peut supposer que π est un isomorphisme au-dessus de X^{reg} et plus précisément, que π consiste en un nombre fini de blowups avec des centres lisses (situés au-dessus de X^{sing}). Plus généralement, on peut considérer une paire (X, D) où X est une variété et D est un diviseur de Weil. Une résolution logarithmique de (X, D) est un morphisme birationnel propre $\pi : Y \to X$ tel que Y est lisse, l'ensemble exceptionnel $\mathrm{Exc}(\pi)$ est de codimension un et $D' := \mathrm{supp}(\pi^{-1}(D) + \mathrm{Exc}(\pi))$

est un simple croisement normal (i.e. $\forall p \in Y$, il existe des coordonnées locales $x_1, \cdots, x_n$ telles que $D' = (x_1 \cdots x_k = 0)$ près de p).

Espaces normaux et de Gorenstein

Un espace analytique complexe X est *normal* si pour chaque sous-ensemble ouvert $U \subset X$, toute fonction méromorphe dans U qui est holomorphe sur U^{reg} s'étend en une fonction holomorphe sur U. Les espaces normaux sont considérés comme ayant des singularités plus "simples" que les espaces analytiques généraux ; en effet, la normalité implique que le lieu singulier est de codimension au moins deux dans X. Nous renvoyons à [Dem12, Ch. 2, § 7] pour une présentation détaillée du concept.

Sur une variété normale X, le faisceau canonique est défini comme $\omega_X = j_* \omega_{X^{\mathrm{reg}}}$ où $j : X^{\mathrm{reg}} \hookrightarrow X$ est l'inclusion naturelle. Une variété normale X est appelée

- 1-*Gorenstein* si ω_X est localement libre;
- $\mathbb{Q}$-*Gorenstein* si, pour tout $x \in X$, il existe $q_x \in \mathbb{N}$ tel que le faisceau pluricanonique $\omega_X^{[q_x]} := j_* \omega_{X^{\mathrm{reg}}}^{q_x}$ est localement libre près de x;
- m-Gorenstein ($m \in \mathbb{N}$) si $\omega_X^{[m]}$ est localement libre sur X.

Si X est projective, on peut trouver $K_{X^{\mathrm{reg}}}$ un diviseur associé au fibré canonique de X^{reg}. Un diviseur canonique de X est un diviseur de Weil obtenu en prenant la fermeture de $K_{X^{\mathrm{reg}}}$ dans X. Dans ce cas, si X est 1-Gorenstein (resp. $\mathbb{Q}$-Gorenstein, m-Gorenstein), alors il est équivalent de dire qu'un diviseur canonique K_X est de Cartier (resp. $\mathbb{Q}$-Cartier, m-Cartier).

Singularités canoniques

Soient X une variété m-Gorenstein pour $m \in \mathbb{N}$ et $\pi : \widetilde{X} \to X$ une résolution de singularités. La *discrépance* est une quantité qui porte le plus grand ordre des pôles d'un générateur local de $\omega_X^{[m]}$ sur la désingularisation $\widetilde{X}$. Plus généralement, on peut aussi considérer une résolution logarithmique $\pi : (Y, \Delta') \to (X, \Delta)$. Précisément, la terminologie des singularités "faibles" est la suivante

Définition 1.2.1. Soit (X, Δ) une paire où X est une variété normale et $\Delta = \sum a_j D_j$ est une somme de diviseurs premiers distincts. Supposons que $\omega_X^{[m]} \otimes \mathcal{O}_X(m\Delta)$ soit localement libre pour un certain $m \in \mathbb{N}$. Soit $\pi : Y \to X$ une résolution logarithmique de (X, Δ). Soient E_i une composante irréductible de $\mathrm{Exc}(\pi)$ et $e \in E_i$ un point générique de E_i. Soit $(y_1, \cdots, y_n)$ un système de coordonnées locales en $e \in Y$ tel que $E_i = (y_1 = 0)$. Alors localement près de e,

$$\begin{aligned}&\pi^*(\text{générateur local de } \omega_X^{[m]} \otimes \mathcal{O}(m\Delta) \text{ à } f(e))\\&= y_1^{c(E_i, X, \Delta)} \cdot (\text{unité}) \cdot (\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_n)^{\otimes m} \otimes (\text{générateur local de } \mathcal{O}_Y(m\pi_*^{-1}(\Delta)))\end{aligned}$$

pour un entier quelconque $c(E_i, X, \Delta)$, où $\pi_*^{-1}(\Delta)$ est la transformée stricte de Δ. Le nombre rationnel $a(E_i, X, \Delta) := \frac{1}{m}c(E_i, X, \Delta)$ est appelé la *discrépance* de E_i par rapport à (X, Δ). Il est indépendant de m. La *discrépance totale* est définie comme suit

$$\operatorname{discrep}(X, \Delta) = \inf_i a(E_i, X, \Delta).$$

Nous disons que (X, Δ) est

$$\left.\begin{array}{l}\text{terminale}\\ \text{canonique}\\ \text{Kawamata log terminale (klt)}\\ \text{purement log terminale (plt)}\\ \text{log canonique (lc)}\end{array}\right\} \text{si } \operatorname{discrep}(X, \Delta) \begin{cases} > 0, \\ \geq 0, \\ > -1 \text{ et } \lfloor \Delta \rfloor \leq 0, \\ > -1, \\ \geq -1. \end{cases}$$

Notez que les discrépances peuvent dépendre du choix d'une résolution logarithmique $\pi : Y \to X$, mais ces notions de singularité ne dépendent pas de π.

De plus, si X est projectif, être terminale (resp. canonique, klt, lc) est équivalent à la description suivante :

$$K_{\widetilde{X}} + \pi_*^{-1}(\Delta) \sim \pi^*(K_X + \Delta) + \sum_i a_i E_i \quad \text{avec } \inf_i a_i > 0 \text{ (resp. } \geq 0, > -1, \geq -1).$$

Exemples

Nous rappelons des exemples explicites de singularités A_k en dimension deux. Ce type de singularités est donné par le quotient de l'action de $\mathbb{Z}_{k+1}$ sur $\mathbb{C}^2$ avec

$$\bar{1} \cdot (a, b) = (e^{\frac{2\pi i}{k+1}} a, e^{-\frac{2\pi i}{k+1}} b).$$

Les monômes $\{a^{k+1}, b^{k+1}, ab\}$ sont des fonctions invariantes $\mathbb{Z}_{k+1}$ sur $\mathbb{C}^2$, et ils induisent une application $\mathbb{C}^2 \ni (a, b) \mapsto (a^{k+1}, b^{k+1}, ab) \in \mathbb{C}^3$ qui identifie l'espace quotient $\mathbb{C}^2/\mathbb{Z}_{k+1}$ et l'hypersurface singulière $(xy - z^{k+1} = 0)$ dans $\mathbb{C}^3$.

Nous calculons maintenant la discrépance dans le cas $k = 1$. Par un changement de coordonnées, la singularité A_1 est isomorphe à $(x^2 + y^2 + z^2 = 0) =: X$ dans $\mathbb{C}^3$, noté $f := x^2 + y^2 + z^2$, qui est irréductible. Pour vérifier que X est normale, on considère $g \in \mathcal{M}_{X,0}$ un germe de fonction méromorphe à 0 tel que g est holomorphe en dehors de l'origine, et on voudrait montrer que $g \in \mathcal{O}_{X,0}$. Par définition, $g = h_1/h_2$ pour certaines fonctions holomorphes $(h_i)_{i=1,2}$ définies près de l'origine, et h_2 est partout non nulle sur X^{reg}. L'irréductibilité de X implique que $(h_2 = 0) \cap X$ est soit de dimension un, soit vide ; par conséquent, h_2 est partout non nulle près de l'origine et alors $g \in \mathcal{O}_{X,0}$. Nous prenons les trois atlas suivants qui couvrent X^{reg} : $U_1 := \{\frac{\partial f}{\partial x} \neq 0\}$, $U_2 = \{\frac{\partial f}{\partial y} \neq 0\}$ et $U_3 = \{\frac{\partial f}{\partial z} \neq 0\}$. Considérons

$$\Omega = \begin{cases} \frac{\mathrm{d}y \wedge \mathrm{d}z}{\frac{\partial f}{\partial x}} = \frac{\mathrm{d}y \wedge \mathrm{d}z}{2x} & \text{sur } U_1 = \{x \neq 0\}; \\ \frac{\mathrm{d}z \wedge \mathrm{d}x}{\frac{\partial f}{\partial y}} = \frac{\mathrm{d}z \wedge \mathrm{d}x}{2y} & \text{sur } U_2 = \{y \neq 0\}; \\ \frac{\mathrm{d}x \wedge \mathrm{d}y}{\frac{\partial f}{\partial z}} = \frac{\mathrm{d}x \wedge \mathrm{d}y}{2z} & \text{sur } U_3 = \{z \neq 0\}. \end{cases}$$

On peut voir que Ω est bien définie et partout non nulle sur X^{reg} ; ainsi, Ω est générateur global de $\omega_{X^{\text{reg}}}$. Alors nous avons $\omega_{X^{\text{reg}}} \simeq \mathcal{O}_{X^{\text{reg}}}$. Par normalité de X, on peut déduire $\omega_X := j_*\omega_{X^{\text{reg}}} \simeq j_*\mathcal{O}_{X^{\text{reg}}} \simeq \mathcal{O}_X$ où $j : X^{\text{reg}} \hookrightarrow X$ est l'inclusion naturelle. Ceci implique que X est 1-Gorenstein. Pour calculer la discrépance de $0 \in X$, nous tirons en arrière Ω vers une résolution :

$$\begin{array}{ccc} \widetilde{X} := p_*^{-1}(X) & \hookrightarrow & \mathrm{Bl}_0(\mathbb{C}^3) \\ \downarrow p & & \downarrow p \\ X = (f = 0) & \hookrightarrow & \mathbb{C}^3 \end{array}$$

où $\mathrm{Bl}_0(\mathbb{C}^3) = \{(w, [\zeta]) \in \mathbb{C}^3 \times \mathbb{CP}^2 \,|\, w \in \zeta\}$ est l'éclatement de $\mathbb{C}^3$ à l'origine. Choisissons l'atlas affine $V_i = \{((w_1, w_2, w_3), [\zeta_1, \zeta_2, \zeta_3]) \in \mathbb{C}^3 \times \mathbb{CP}^2 \,|\, \zeta_i \neq 0\}$ pour $i \in \{1, 2, 3\}$. Sur V_1, (w_1, v_2, v_3) forme un système de coordonnées, où $v_i = \zeta_i/\zeta_1$ et $w_i = w_1 v_i$. On a $\widetilde{X} \cap V_1 = (1 + v_2^2 + v_3^2 = 0)$ qui est lisse. Nous posons

$$W_{1,2} = \left\{ (w_1, v_2, v_3) \in \widetilde{X} \cap V_1 \,\middle|\, v_2 \neq 0 \right\}, \text{ et } W_{1,3} = \left\{ (w_1, v_2, v_3) \in \widetilde{X} \cap V_1 \,\middle|\, v_3 \neq 0 \right\}.$$

Alors nous avons

$$p^*\Omega_{|W_{1,2}} = \frac{\mathrm{d}(w_1 v_3) \wedge \mathrm{d}w_1}{2 w_1 v_2} = \frac{\mathrm{d}v_3 \wedge \mathrm{d}w_1}{2 v_2}$$

et de même $p^*\Omega_{|W_{1,3}} = \frac{\mathrm{d}w_1 \wedge \mathrm{d}v_2}{2v_3}$. De façon similaire sur les autres atlas, nous concluons que la discrépance de $0 \in X = (x^2 + y^2 + z^2 = 0)$ est nulle, donc c'est une singularité canonique.

Nous extrayons certains faits de [KM98, Ch. 4] et [Kol13, Ch. 3] (voir aussi [EGZ09]) :

Fait 1.2.1. *Supposons que S est une surface algébrique normale, p est une singularité isolée. Alors*

- *p est canonique $\iff$ près de p, S est localement isomorphe à $\mathbb{C}^2/G$ où $G < \mathrm{SL}(2, \mathbb{C})$ est fini ;*
- *p est klt $\iff$ près de p, S est localement isomorphe à $\mathbb{C}^2/G$ où $G < \mathrm{GL}(2, \mathbb{C})$ est fini.*

Fait 1.2.2. *En dimension supérieure, les singularités quotient sont encore klt. Fixons $n \geq 2$ et soit $H \subset \mathbb{CP}^{n+1}$ une hypersurface lisse de degré d. Le cône affine sur H n'a que des singularités canoniques si et seulement si $d \leq n + 1$. En particulier, le point double ordinaire $x^2 + y^2 + z^2 + w^2 = 0$ a une singularité canonique à l'origine, mais ce n'est pas une singularité quotient.*

Pour plus d'informations sur les singularités et la MMP, nous recommandons au lecteur de suivre [KM98, Kol13].

1.2.2 Un récapitulatif sur les courants positifs et les fonctions plurisousharmoniques

Maintenant, nous rappelons quelques définitions de base concernant les fonctions plurisousharmoniques et les opérateurs de ces fonctions. Pour une présentation exhaustive, nous renvoyons le lecteur à [Dem12, GZ17]. Nous définissons l'opérateur réel $\mathrm{d}^c = \frac{\mathrm{i}}{2}(\bar{\partial} - \partial)$ de sorte que $\mathrm{dd}^c = \mathrm{i}\partial\bar{\partial}$.

Courants positifs

Un *courant* T de degré p, sur une variété réelle lisse M de dimension m, est une fonctionnelle linéaire sur l'espace des $(m-p)$-formes lisses à support compact qui est continue par rapport à la semi-norme $\mathcal{C}_c^\infty$. Nous désignons par $\langle T, \theta \rangle$ le couplage d'un courant de degré p et d'une $(m-p)$-forme lisse à support compact. Notez qu'un courant de degré p peut être vu comme une forme de degré p avec des coefficients distribution. Le courant $\mathrm{d}T$ est le courant de degré $(p+1)$ défini par

$$\langle \mathrm{d}T, \theta \rangle = (-1)^{m-p+1} \langle T, \mathrm{d}\theta \rangle$$

pour toute $(p-1)$-forme d'essai θ ; on dit que T est fermé si $\mathrm{d}T = 0$.

Maintenant, nous supposons que X est une variété complexe lisse de dimension complexe n. Une k-forme différentielle peut être décomposée en formes de bidegré pur (p,q) avec $p+q=k$. Une (p,q)-forme s'écrit localement θ comme $\theta = \sum_{|I|=p,|J|=q} \theta_{I\bar{J}} \mathrm{d}z_I \wedge \mathrm{d}\bar{z}_J$ où $I := \{1 \le i_1 < \cdots < i_p \le n\}$ (resp. $J := \{1 \le j_1 < \cdots < j_q \le n\}$) est un multi-index de longueur p (resp. q), et $\mathrm{d}z_I = \mathrm{d}z_{i_1} \wedge \cdots \wedge \mathrm{d}z_{i_p}$ (resp. $\mathrm{d}\bar{z}_J = \mathrm{d}\bar{z}_{j_1} \wedge \cdots \wedge \mathrm{d}\bar{z}_{j_q}$). Alors un courant de bidegré (p,q) est une fonctionnelle linéaire continue sur l'espace des formes de bidegré $(n-p, n-q)$ lisses à support compact.

Un exemple intéressant d'un (p,p)-courant fermé est le courant d'intégration sur un sous-ensemble analytique A de dimension $(n-p)$, qui est noté $[A]$ et défini comme $\langle [A], \theta \rangle := \int_{A^{\mathrm{reg}}} \theta$ pour toute forme test θ. Lelong [Lel57] a prouvé que le courant $[A^{\mathrm{reg}}]$ sur $X \setminus A^{\mathrm{sing}}$ a une masse finie dans un voisinage de chaque point $z_0 \in A^{\mathrm{sing}}$, et son extension triviale $[A]$ est un courant fermé (positif) sur X. Dans la suite, nous nous intéressons particulièrement aux courants de bidegré $(1,1)$. Localement, un $(1,1)$-courant peut être exprimé comme suit

$$T = \sum_{1 \le j,k \le n} T_{j\bar{k}} \mathrm{i} \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k,$$

où les $T_{i\bar{j}}$ sont des distributions. Un $(1,1)$-courant est dit positif si la distribution $\sum_{j,k} a_j \bar{a}_k T_{j\bar{k}}$ est positive pour chaque $a \in \mathbb{C}^n$. Si c'est le cas, pour chaque (j,k), la distribution $T_{j\bar{k}}$ est une mesure de Radon complexe.

Fonctions plurisousharmoniques

Soit Ω un domaine dans $\mathbb{C}^n$. Une fonction semi-continue supérieure $u : \Omega \to \mathbb{R} \cup \{-\infty\}$ est dite *pluriousharmonique* (psh en abrégé) si pour chaque $x \in \Omega$, pour tout $\zeta \in B_1(0) \subset \mathbb{C}^n$ et pour tout $r < \mathrm{dist}(x, \partial\Omega)$,

$$u(z) \leq \frac{1}{2\pi} \int_0^{2\pi} u(x + r\zeta e^{\mathrm{i}\theta}) \mathrm{d}\theta.$$

En particulier, u est une fonction sousharmonique le long de toute ligne complexe. Notons par $\mathrm{PSH}(\Omega)$ l'espace des fonctions psh sur Ω qui ne sont pas identiquement $-\infty$. Un exemple prototypique d'une fonction psh est $\log|f|$ où f est une fonction holomorphe non nulle sur Ω. Dans ce cas, $\frac{1}{\pi}\mathrm{dd^c} \log|f|$ est un $(1,1)$-courant positif fermé qui est exactement le courant d'intégration le long de $Z_f := (f = 0)$ par la formule de Poincaré–Lelong.

La définition peut encore être étendue aux variétés complexes lisses. Une fonction est dite *quasi-plurisousharmonique* (qpsh) si elle peut être localement écrite comme la somme d'une fonction lisse et d'une fonction psh. Sur une variété complexe lisse X avec une $(1,1)$-forme lisse ω, une fonction u est ω-plurisousharmonique (ω-psh) si elle est qpsh et $\omega + \mathrm{dd^c}u \geq 0$ au sens des courants. Nous désignons par $\mathrm{PSH}(X, \omega)$ l'ensemble de toutes les fonctions ω-psh sur la variété complexe lisse X.

Nombres de Lelong

Un invariant important pour étudier les singularités des fonctions psh est le nombre de Lelong. Soit u une fonction psh sur un domaine $\Omega \subset \mathbb{C}^n$. Le nombre de Lelong de u en un point $x \in \Omega$ est défini par

$$\nu(u, x) = \sup \{\gamma \geq 0 \,|\, u(z) \leq \gamma \log|z - x| + O(1) \text{ près de } x\}.$$

En particulier, si $u = \log|f|$ pour une certaine fonction holomorphe f, alors le nombre de Lelong $\nu(u, x)$ est l'ordre d'annulation de f en x. Il existe également une définition équivalente :

$$\nu(u, x) = \lim_{r \to 0}{}^{\downarrow} \frac{\int_{B_r(x)} \mathrm{dd^c}u \wedge (\mathrm{dd^c}|z|^2)^{n-1}}{(2\pi r^2)^{n-1}}$$

qui est aussi appelée la masse projective. Cette définition permet de généraliser le nombre de Lelong aux courants de bidegré (p, p) positifs et c'est également ce que nous utilisons dans le chapitre 4.

Opérateurs de Monge–Ampère complexes

Un rôle important dans l'analyse complexe est joué par la théorie de Bedford–Taylor : [BT76, BT82]. Fixons un domaine $\Omega \subset \mathbb{C}^n$. Soit T un courant positif fermé de bidegré (p, p) et soit u une fonction psh localement bornée sur Ω. Le courant T a des coefficients de mesure, donc uT

est un courant bien défini d'ordre zéro ; on définit alors

$$dd^c u \wedge T = dd^c(uT).$$

Le résultat suivant est crucial pour travailler avec des fonctions psh qui ne sont que localement bornées :

Théorème (Bedford–Taylor [BT82]). *Soit $(u_j)_j$ une suite de fonctions psh localement bornées décroissante en $u \in \mathrm{PSH}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$. Alors*

$$dd^c u_j \wedge T \to dd^c u \wedge T$$

dans le sens des courants sur Ω. En particulier, $dd^c u \wedge T$ est un courant positif fermé ; par conséquent, ses coefficients sont également des mesures de Radon complexes (distributions d'ordre zéro) sur Ω.

Par récurrence, on peut alors montrer que la mesure de Monge–Ampère complexe $(dd^c u)^n$ a un sens pour les fonctions psh localement bornées dans Ω. Sur une variété kählérienne compacte lisse (X, ω) de dimension n, la notion de courants positifs étant purement local, pour une fonction u bornée de ω-psh, il est naturel de définir sa mesure de Monge–Ampère complexe comme $(\omega + dd^c u)^n = (dd^c(\rho + u))^n$, où ρ est un potentiel local lisse de ω (c-à-d que $dd^c \rho = \omega$). Sur une variété hermitienne compacte lisse (X, ω), il est un peu plus compliqué de définir la mesure de Monge–Ampère. Nous fixons d'abord une fonction locale lisse strictement psh ρ qui satisfait $0 < C^{-1} dd^c \rho \le \omega \le C dd^c \rho$ pour une certaine constante $C \ge 1$. Nous définissons ensuite

$$\begin{aligned}(\omega + dd^c u)^n &= ((\omega - dd^c \rho) + dd^c(\rho + u))^n \\ &= \sum_{k=0}^{n} \binom{n}{k} (\omega - dd^c \rho)^{n-k} \wedge (dd^c(\rho + u))^k .\end{aligned}$$

On peut vérifier que cette définition ne dépend pas du choix du potentiel local ρ.

Formes et courants sur des espaces singuliers

Soit X un espace analytique réduit. Notons $\mathcal{C}^k_{p,q}(X)$ l'espace des (p,q)-formes de classe $\mathcal{C}^k$ qui sont définies comme suit : supposons que U est un sous-ensemble ouvert de X et que $j : U \to \Omega$ soit un plongement de U dans un sous-ensemble ouvert $\Omega \subset \mathbb{C}^N$. Nous notons par $\mathcal{C}^k_{p,q}(U)$ l'image du morphisme de restriction

$$j^* : \mathcal{C}^k_{p,q}(\Omega) \to \mathcal{C}^k_{p,q}(U^{\mathrm{reg}})$$

munie de la topologie quotient. On peut vérifier que cette notion est indépendante du choix du plongement j (voir [Dem85, p. 14]).

Notons par $\mathcal{D}_{p,q}(X)$ (resp. $\mathcal{D}^k_{p,q}(X)$) l'espace des $\mathcal{C}^\infty$ (resp. $\mathcal{C}^k$) formes (p,q) à support compact sur X munie de la topologie induite par les semi-normes. L'espace dual, $(\mathcal{D}_{p,q}(X))'$, est par définition l'espace des courants de bidegré $(n-p, n-q)$ sur X. Précisément, pour tout courant $T \in (\mathcal{D}_{p,q}(X))'$ et tout plongement local $j : U \to \Omega$, il existe un courant $j_*T \in (\mathcal{D}_{p,q}(\Omega))'$ avec

$$\langle j_*T, v\rangle = \langle T, j^*v\rangle$$

pour toute forme $v \in \mathcal{D}_{p,q}(\Omega)$ et le support de j_*T est contenu dans $j(U)$.

Tous les objets passés en revue dans cette section (fonctions psh, nombres de Lelong, mesures de MA), peuvent être définis sur des espaces complexes. Pour plus de détails et une introduction rapide, nous renvoyons le lecteur à [Dem85] et à la section 4.1. Soulignons que les deux définitions du nombre de Lelong (pôle logarithmique et masse projective) ne coïncident pas nécessairement sur un espace complexe singulier (voir [BBE+19, Rmk. A.5]).

1.2.3 Familles de variétés et quelques remarques

Dans cette section, nous passons en revue certains faits généraux concernant les familles de variétés faiblement singulières. Nous considérons habituellement une famille de variétés $\pi : \mathcal{X} \to \mathbb{D}$ comme dans le cadre suivant :

Cadre 1.2.2 (Cadre géométrique)**.** Soit $\mathcal{X}$ une variété de dimension $(n+1)$. Supposons que $\pi : \mathcal{X} \to \mathbb{D}$ est une application holomorphe surjective propre dont les fibres (schématiques) connectées $X_t = \pi^{-1}(t)$ sont toujours des variétés (irréductibles, réduites).

Les propriétés suivantes proviennent alors automatiquement du cadre géométrique 1.2.2 :

- Puisque nous supposons que $\mathcal{X}$ est irréductible, nous notons d'abord que X_t est de codimension un (cf. [Dem12, Ch. II, Thm. 6.2]).
- Puisque π est ouvert (cf. [Dem12, Ch. II, Thm. 5.7]) et que la base $\mathbb{D}$ est lisse, la famille $\pi : \mathcal{X} \to \mathbb{D}$ est plate (cf. [BS76, Ch. V, Thm. 2.13]).

Rappelons qu'une application $f : X \to Y$ est plate si la carte induite sur chaque tige $f_p : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$ est une application plate d'anneaux ; autrement dit, $\mathcal{O}_{X,p}$ est plat en tant que module $\mathcal{O}_{Y,f(p)}$. Un R-module M est plat si, chaque fois que $N_1 \to N_2 \to N_3$ est une suite exacte de R-modules, la séquence $M \otimes_R N_1 \to M \otimes_R N_2 \to M \otimes_R N_3$ est exacte sur R. Il est difficile d'expliquer l'intuition géométrique de la platitude ; voici une citation de Mumford [Mum88, p. 295] : "The concept of flatness is a riddle that comes out of algebra, but which technically is the answer to many prayers."

En ajoutant d'autres hypothèses sur la fibre centrale X_0 et les espaces totaux $\mathcal{X}$, on peut obtenir d'autres propriétés :

Fait 1.2.3. *D'après [Gro66, Thm. 12.2.1 (v)], la normalité est ouverte sur la base $\mathbb{D}$ si l'application π est plate ; à savoir, X_t est normale pour tous les t suffisamment proches de zéro. D'autre part, si X_t est* *normale pour tout $t \in \mathbb{D}$, alors $\mathcal{X}$ l'est aussi par [Gro65, Cor. 5.12.7]. Par conséquent, à rétrécissement de $\mathbb{D}$ près, $\mathcal{X}$ et $(X_t)_{t\in\mathbb{D}}$ sont normales.*

Fait 1.2.4. *Supposons que X_0 est normale et que $\mathcal{X}$ est 1-Gorenstein (resp. $\mathbb{Q}$-Gorenstein). Si X_0 a des singularités canoniques (resp. klt), par inversion de l'adjonction (cf. [Kol13, Thm. 4.9]), $\mathcal{X}$ a des singularités canoniques (resp. klt) près de X_0. De plus, X_t possède des singularités canoniques (resp. klt) pour tout t proche de 0 (cf. [Kol13, Cor. 4.10]).*

Fait 1.2.5. *Supposons que X_0 est Calabi–Yau (c-à-d que ω_{X_0} est trivial) et que $\mathcal{X}$ est $\mathbb{Q}$-Gorenstein. Alors les éléments suivants sont équivalents*

- *$\omega_{\mathcal{X}/\mathbb{D}}$ (ou $\omega_{\mathcal{X}}$) est trivial jusqu'à la réduction de $\mathbb{D}$;*
- *ω_{X_t} est trivial pour tout t petit (c-à-d que X_t est Calabi–Yau pour t suffisamment proche de 0).*

Il en va de même pour $\mathbb{Q}$-Calabi–Yau (c-à-d que $\omega_{X_0}^{[q]}$ est trivial pour un certain $q \in \mathbb{N}$) avec une puissance positive uniforme q. Ceci est une conséquence du théorème de l'image directe de Grauert (voir la preuve comme dans le lemme 4.6.1).

Lissage

Une famille $\pi : \mathcal{X} \to \mathbb{D}$ est appelée un *lissage* d'une variété X si la famille $\pi : \mathcal{X} \to \mathbb{D}$ satisfait

- $\mathcal{X}$ est lisse sur $\mathbb{D} \setminus \{0\}$;
- la fibre centrale (schématique) X_0 est isomorphe à X.

Si une variété X admet un lissage, alors on l'appelle une variété lissable. Les variétés lissables peuvent être considérées comme des éléments situés sur les limites "bien conformées" d'un espace modulaire de variétés lisses. Nous donnons un exemple simple comme suit :

$$\mathcal{X} = \left\{([z_1 : z_2 : z_3 : z_4], t) \in \mathbb{CP}^3 \times \mathbb{D} \,\middle|\, z_1^2 + z_2^2 + z_3^2 = t z_4^2\right\} \subset \mathbb{CP}^3 \times \mathbb{D}.$$

L'application $\pi : \mathcal{X} \to \mathbb{D}$ est la projection sur $\mathbb{D}$. On peut vérifier que l'espace total $\mathcal{X}$ est lisse et que chaque fibre est également lisse, sauf la fibre centrale.

1.3 Présentation des résultats

1.3.1 Métriques de Gauduchon singulières

Les métriques de Gauduchon, introduites pour la première fois par Gauduchon [Gau77], sont des généralisations utiles des métriques kählériennes sur une variété non-kählérienne lisse.

Théorème (Gauduchon [Gau77])**.** *Soit X une variété complexe compacte lisse de dimension n. Fixons une métrique hermitienne ω sur X. Il existe une fonction lisse positive $\rho \in \mathcal{C}^\infty(X)$ telle que*

$$dd^c(\rho\omega^{n-1}) = 0.$$

De plus, ρ est unique à un multiple constant positif près.

Une application importante de l'existence des métriques de Gauduchon a été mise en évidence par l'extension de la correspondance Donaldson–Uhlenbeck–Yau (DUY) [Don85, UY86, Don87] aux variétés non-kählériennes lisses. La correspondance de DUY stipule que, sur une variété kählérienne compacte (X, ω), un fibré vectoriel holomorphe E est polystable en pente si et seulement si le fibré E admet une métrique de Hermite–Einstein. Elle peut être considérée comme une généralisation de la métrique de Kähler–Einstein sur le fibré tangent. Grâce à Li et Yau [LY87], la correspondance DUY tient par rapport à une métrique de Gauduchon. La version non-kählérienne de la correspondance DUY est très utile pour classer les surfaces de classe VII avec des méthodes de théorie de la jauge (voir [Tel19] et les références qui s'y rapportent).

Sur la base des considérations des sections précédentes, il est donc intéressant de s'interroger sur la correspondance DUY sur les espaces singuliers et les fibrés vectoriels singuliers. Bando et Siu [BS94] ont prouvé que sur les variétés de Kähler compactes, le théorème de DUY est valable non seulement pour les fibrés vectoriels mais aussi pour les faisceaux réflexifs. Très récemment, le théorème de DUY sur les espaces kählériens normaux singuliers a été étudié par de nombreux auteurs (voir [CW21, Ou22, Che22]) dans différents contextes. Il est également naturel de se demander si le théorème de DUY est valable pour les variétés non-kählériennes singulières. Une version singulière de la métrique de Gauduchon est alors nécessaire pour définir la notion de stabilité des fibrés vectoriels ou même des faisceaux réflexifs sur un espace non-kählérien singulier.

Nous présentons ici plusieurs résultats importants qui apparaîtront dans Chapitres 2 et 3. Nous établissons d'abord une inégalité de type Harnack des facteurs de Gauduchon :

Théorème A (= Thm. 2.2.1+3.1.1)**.** *Soit (X, ω) une variété hermitienne compacte. Si $\rho \in \mathcal{C}^\infty(X^{\mathrm{reg}})$ est une fonction strictement positive avec $\inf_{X^{\mathrm{reg}}} \rho = 1$ qui satisfait $\mathrm{dd^c}(\rho\omega^{n-1}) = 0$ sur X^{reg}, alors il existe une constante uniforme $C_G(n, \omega, X)$ telle que*

$$\sup_{X^{\mathrm{reg}}} \rho \leq C_G.$$

Dans le cas lisse, il n'y a pas d'obstruction géométrique à l'existence des métriques de Gauduchon, sauf la compacité. Nous conjecturons qu'un énoncé similaire pourrait s'appliquer à une variété hermitienne compacte ; à savoir, étant donné une variété hermitienne compacte (X, ω), on peut trouver une métrique de Gauduchon bornée (cf. Definition 2.1.3) qui est conformalement équivalente à ω. Nous proposons quelques classes de variétés pour lesquelles la conjecture tient. Plus précisément, considérons l'hypothèse suivante.

Hypothèse (G). *Nous supposons que la variété singulière compacte X satisfait l'une des conditions suivantes*

G.1 *X est lissable ;*

G.2 *X admet une résolution de singularités $\mu : \widetilde{X} \to X$ qui est donnée par une composition d'un nombre fini d'éclatements de points. En d'autres termes, μ peut être décomposé en*

$$\widetilde{X} = X_m \xrightarrow{p_m} X_{m-1} \xrightarrow{p_{m-1}} \cdots \xrightarrow{p_2} X_1 \xrightarrow{p_1} X_0 = X$$

et pour chaque $k \in \{1, \cdots, m\}$, $p_k : X_k \to X_{k-1}$ est un éclatement de points dans X_{k-1}^{sing}.

Remarque 1.3.1. Les cas suivants remplissent la condition **G.2** :

(i) X est une surface de Gorenstein $\mathbb{Q}$ ($\dim_{\mathbb{C}} X = 2$) ;

(ii) Les singularités de X sont des points doubles ordinaires (ODP). En particulier, les singularités sont localement isomorphes à la quadrique $Q = \left\{ z \in \mathbb{C}^{n+1} \,\middle|\, \sum_{j=1}^{n+1} z_j^2 = 0 \right\}$.

En effet, un éclatement avec un centre lisse d'une variété de Gorenstein $\mathbb{Q}$ est toujours de Gorenstein $\mathbb{Q}$; donc en dimension deux, leurs singularités sont juste des points par normalité. D'autre part, les singularités ODP peuvent être résolues en ne prenant qu'un de point.

Sous l'hypothèse (G), nous obtenons la version singulière suivante du théorème de Gauduchon.

Théorème B (= Thm. 2.0.1+3.0.1)**.** *Supposons que X est une variété complexe compacte de dimension n. De plus, supposons que X remplit l'hypothèse (G). Soit ω une métrique hermitienne sur X. Alors il existe une fonction bornée $\rho \in \mathcal{C}^\infty(X^{\text{reg}})$ avec $\inf_{X^{\text{reg}}} \rho = 1$ telle que $\mathrm{dd^c}(\rho\omega^{n-1}) = 0$. En particulier, sur X^{reg}, $\omega_G := \rho^{\frac{1}{n-1}}\omega$ définit une métrique de Gauduchon bornée qui est conformalement équivalente à ω.*

Dans le théorème de Gauduchon, à un multiple positif constant près, il existe une unique métriques de Gauduchon dans une classe hermitienne conforme. Nous étendons le résultat de l'unicité aux métriques de Gauduchon bornées conformes équivalentes. Nous montrons également une propriété de l'extension en tant qu'un courant plurifermé (i.e. $\mathrm{dd^c}$-fermé).

Théorème C (= Thm. 2.0.2)**.** *Supposons que X soit une variété de dimension n dotée d'une métrique de Gauduchon bornée ω_G. Nous avons alors les propriétés d'unicité et d'extension suivantes.*

(i) Si ω'_G est une autre métrique de Gauduchon bornée dans la classe conforme de ω_G, alors ω'_G doit être un multiple positif de ω_G ;

(ii) Soit T le $(n-1, n-1)$-courant positif obtenu comme l'extension triviale de ω_G^{n-1}. Alors T est un courant plurifermé.

1.3.2 Familles de métriques de Chern–Ricci plates

Comme mentionné dans la section 1.1.4, il est important d'étudier les métriques canoniques singulières sur les variétés singulières et les familles dégénérées. La solution de la conjecture (singulière) de Calabi [Yau78, EGZ09] fournit un théorème d'existence très puissant pour les métriques de Kähler–Einstein à courbure de Ricci négative ou nulle. Une telle métrique est une véritable métrique de Kähler–Einstein sur le lieu lisse et possède des potentiels "bornés" près des singularités. Sur une variété de Fano singulière, Li [Li22] (voir aussi [LTW21]) a récemment montré que la correspondance de Chen–Donaldson–Sun est toujours valable.

Dans une famille $\pi : \mathcal{X} \to \mathcal{Y}$, une situation importante est celle où la fibre X_y est dotée d'une métrique de Kähler–Einstein. Pour construire une métrique de Kähler–Einstein (singulière), l'étape la plus cruciale et la plus difficile est d'obtenir une estimation uniforme des solutions aux équations de Monge–Ampère complexes dégénérées. À partir du résultat fondamental de Kołodziej [Koł98] sur les équations de Monge–Ampères complexes dégénérées, la théorie du pluripotentiel est devenue un outil puissant pour étudier les métriques de Kähler–Einstein singulières. Pour étudier les familles de métriques de Kähler–Einstein singulières, Di Nezza, Guedj, et Guenancia [DGG20] ont développé les premiers pas de la théorie du pluripotentiel dans les familles de variétés kählériennes. On peut alors s'intéresser au fait qu'une déclaration similaire s'applique à des familles de variétés qui pourraient être non-kählériennes.

Dans la suite, nous considérons toujours une famille $\pi : \mathcal{X} \to \mathbb{D}$ qui satisfait le cadre 1.2.2. Soit ω une métrique hermitienne ; elle induit une métrique hermitienne $\omega_t = \omega_{|X_t}$ pour chaque $t \in \mathbb{D}$. Nous prouvons une estimation uniforme de L^∞ basée sur une conjecture de comparaison sup-L^1 (Conjecture (SL)) qui sera expliquée plus tard.

Théorème D (= Thm. 4.0.1)**.** *Soit $\pi : (\mathcal{X}, \omega) \to \mathbb{D}$ une famille de variétés hermitiennes compactes, localement irréductibles et $f_t \in L^p(X_t, \omega_t^n)$ une famille de densités pour un certain $p > 1$. Supposons que $\pi : (\mathcal{X}, \omega) \to \mathbb{D}$ satisfait la conjecture (SL) et que les $(f_t)_{t\in\mathbb{D}}$ satisfont les bornes intégrales suivantes : il existe des constantes positives c_f et C_f telles que $\forall t \in \mathbb{D}$,*

$$c_f < \int_{X_t} f_t^{\frac{1}{n}} \omega_t^n \quad \text{et} \quad \int_{X_t} f_t^p \omega_t^n < C_f.$$

Et que pour chaque $t \in \mathbb{D}$, que la paire (φ_t, c_t) soit une solution à l'équation de Monge–Ampère complexe

$$(\omega_t + dd_t^c \varphi_t)^n = c_t f_t \omega_t^n, \quad \text{et} \quad \sup_{X_t} \varphi_t = 0.$$

Il existe alors une constante $C_{\mathrm{MA}} = C_{\mathrm{MA}}(c_f, C_f, C_{SL}, \mathcal{X}, \omega) > 0$ telle que pour tout $t \in \mathbb{D}_{1/2}$,

$$c_t + c_t^{-1} + \|\varphi_t\|_{L^\infty(X_t)} \leq C_{\mathrm{MA}}.$$

La conjecture de comparaison sup-L^1 suivante est introduite pour la première fois par Di Nezza, Guedj, et Guenancia [DGG20, Conjecture 3.1] dans le cadre de Kähler. Ici, nous énonçons une analogue hermitienne :

Conjecture (SL). *Il existe une constante uniforme $C_{SL} > 0$ telle que pour tout $t \in \mathbb{D}_{1/2}$ et pour tout*

$\psi_t \in \mathrm{PSH}(X_t, \omega_t)$,

$$\sup_{X_t} \psi_t - C_{SL} \leq \frac{1}{V_t} \int_{X_t} \psi_t \omega_t^n,$$

où V_t est le volume de X_t par rapport à la métrique ω_t.

Dans [DGG20], les auteurs ont également fourni de grandes classes de familles d'espaces kählériens où la conjecture tient. Pour traiter les familles projectives et de lissage, ils obtiennent une borne inférieure uniforme des fonctions de Green via une méthode de noyau de chaleur de Cheng et Li [CL81]. Récemment, en établissant des estimations uniformes des noyaux de chaleur sur une variété kählérienne, Ou [Ou22] a résolu la conjecture sur toutes les familles d'espaces kählériens satisfaisant le cadre 1.2.2. On note que l'irréductibilité de chaque fibre est nécessaire ; sinon, il existe un contre-exemple quantitatif donné dans la [DGG20, Example 3.5]. En suivant des idées similaires à celles de [DGG20], nous démontrons la conjecture (SL) avec quelques hypothèses supplémentaires sur les singularités de la famille $\pi : \mathcal{X} \to \mathbb{D}$.

Proposition E (= Prop. 4.0.2)**.** *La conjecture (SL) tient dans l'une des conditions suivantes :*

- *π est localement triviale ;*
- *X_t est lisse pour chaque $t \neq 0$, et X_0 a des singularités isolées.*

Nous appliquons ensuite ce résultat aux "bonnes" familles de variétés de Calabi–Yau non-kählériennes et nous demandons comment le contrôle sur les potentiels de Chern–Ricci varie dans les familles :

Théorème F (= Thm. 4.0.3)**.** *Supposons que $\mathcal{X}$ est normale, $K_{\mathcal{X}}$ est trivial, et $\pi : \mathcal{X} \to \mathbb{D}$ est un lissage de la fibre centrale X_0 dont les singularités sont canoniques et isolées. Soit Ω une trivialisation de $K_{\mathcal{X}/\mathbb{D}}$ et Ω_t la restriction de Ω sur X_t. Pour chaque $t \in \mathbb{D}$, soit $(\varphi_t, c_t) \in (\mathrm{PSH}(X_t, \omega_t) \cap L^\infty(X_t)) \times \mathbb{R}_{>0}$ une paire résolvant l'équation de Monge–Ampère complexe*

$$(\omega_t + dd_t^c \varphi_t)^n = c_t \Omega_t \wedge \overline{\Omega_t} \quad et \quad \sup_{X_t} \varphi_t = 0.$$

Alors il existe une constante uniforme $C_{\mathrm{MA}} > 0$ telle que pour tout $t \in \mathbb{D}_{1/2}$,

$$c_t + c_t^{-1} + \|\varphi_t\|_{L^\infty(X_t)} \leq C_{\mathrm{MA}}.$$

Chapter 2

Singular Gauduchon metrics

In this chapter, we provide (with slight modification) the content of the article [Pan22].

In 1977, Gauduchon proved that on every compact hermitian manifold (X, ω) there exists a conformally equivalent hermitian metric ω_{G} which satisfies $\mathrm{dd^c}\omega_{\mathrm{G}}^{n-1} = 0$. In this note, we extend this result to irreducible compact singular hermitian varieties which admit a smoothing.

Introduction

Let X be an n-dimensional compact complex manifold equipped with a positive-definite smooth $(1,1)$-form ω. We also call ω a hermitian metric because such ω corresponds to a hermitian metric. A famous theorem of Gauduchon [Gau77] says that there exists a metric ω_{G} in the conformal class of ω such that $\mathrm{dd^c}\omega_{\mathrm{G}}^{n-1} = 0$ and the metric ω_{G} is unique up to a positive multiple. These kind of metrics are since then called *Gauduchon metrics*. The conformal factor ρ satisfying $\omega_{\mathrm{G}}^{n-1} = \rho\omega^{n-1}$ is called *Gauduchon factor*.

In complex geometry, finding canonical metrics on complex manifolds is a central problem. Two celebrated examples are Yau's solution of the Calabi conjecture [Yau78] and Uhlenbeck and Yau's characterization of the existence of hermitian–Einstein metrics on stable vector bundles [UY86]. These theorems are established on Kähler manifolds. When the manifold is non-Kähler, the analysis is more difficult because hermitian metrics are no longer closed. In such cases, Gauduchon metrics provide a useful substitute. For instance, Li and Yau [LY87] used Gauduchon metrics to define the slope stability of vector bundles on compact non-Kähler manifolds. As a consequence, they generalized the Uhlenbeck–Yau theorem to non-Kähler setting. For generalized Calabi–Yau type problem in non-Kähler context, Tosatti and Weinkove [TW10] showed that for arbitrary representative $\Psi \in c_1^{\mathrm{BC}}(X)$ of the first Bott–Chern class of X, there exists a hermitian metric ω such that $\mathrm{Ric}(\omega) = \Psi$ by solving complex Monge–Ampère equations. In their proof, Gauduchon metrics play an important role to simplify calculations. Furthermore, Székelyhidi, Tosatti and Weinkove [STW17] proved that one can even find a Gauduchon metric with prescribed Chern–Ricci curvature. On the other hand, Angella, Calamai and

Spotti [ACS17] studied the Chern–Yamabe problem (i.e. finding constant Chern scalar curvature metrics in the conformal class of a given metric ω). They used Gauduchon metrics to define a conformal invariant called the Gauduchon degree and showed that if a metric ω has non-positive Gauduchon degree then the Chern–Yamabe problem admits a solution. For more applications and results about Gauduchon metrics, the interested reader is referred to [FU13, FWW13, Li21] and the references therein.

From an algebraic point of view, singularities are ubiquitous as they occur in various contexts, notably in the minimal model program and moduli theory. Ueno [Uen75] found a birational class of three-dimensional complex manifolds which does not admit a smooth minimal model. In moduli theory, it is necessary to deal with singular varieties when compactifying moduli spaces of smooth manifolds. Already in dimension one, the fundamental domain of moduli space of elliptic curves is non-compact and nodal curves lie on its boundary. On the other hand, in non-Kähler geometry, investigating singular varieties admitting a non-Kähler smoothing is an essential issue due to close interactions of string theory and mathematics established over the past 40 years. In the 1980s, a large class of non-Kähler Calabi–Yau threefolds was built via conifold transitions introduced by Clemens [Cle83] and Friedman [Fri86]. Roughly speaking, the process goes as follows: contracting a collection of disjoint $(-1,-1)$-curves from a Kähler Calabi–Yau threefold X to obtain a singular Calabi–Yau variety X_0 and then smoothing singularities of X_0, one obtains a family of Calabi–Yau threefolds $(X_t)_{t\neq 0}$ which are generally non-Kähler. Thus, it is important to understand the geometric structure on X_t induced by the original Calabi–Yau manifold X. Experts believe that the Hull–Strominger system [Hul86, Str86] provides a natural candidate. These models attracted a lot of attentions in recent years (cf. [Rei87, Fri91, Tia92, Ros06, FY08, Chu12, FLY12, PPZ18, CPY21] and the references therein).

Given these considerations, it is legitimate to look for canonical metrics or special metrics on singular hermitian varieties. In this note, we focus on Gauduchon metrics. A standard way to give a metric structure on a singular complex space is to restrict an ambient metric in local embeddings (see Definition 2.1.1 for the precise definition). Then we address the following question.

Question. Suppose that X is an irreducible, reduced, compact complex space equipped with a hermitian metric ω. Can one find a Gauduchon metric ω_G in the conformal class of ω?

The purpose of this note is to give partial answers in the setup of smoothable singularities. This means that the variety can be approximated by a family of smooth manifolds and the hermitian metric is the restriction of an ambient smooth metric. The precise statements are as follows.

Setup (S). *Let $\mathcal{X}$ be an $(n+1)$-dimensional, irreducible, reduced, complex analytic space. Suppose that*

- $\pi : \mathcal{X} \to \mathbb{D}$ *is a proper, surjective, holomorphic map with connected fibres;*

- *π is smooth over the punctured disk $\mathbb{D}^*$;*
- *the central fibre X_0 is an n-dimensional, irreducible, reduced, compact complex analytic space.*

Let ω be a hermitian metric on $\mathcal{X}$ in the sense of Definition 2.1.1. For each $t \in \mathbb{D}$, we define the hermitian metric ω_t on fibre X_t by restriction (i.e. $\omega_t := \omega|_{X_t}$).

In Setup (S), on each smooth fibre X_t, there exists a Gauduchon factor ρ_t with respect to ω_t by Gauduchon's theorem. We may normalize these Gauduchon factors such that $\inf_{X_t} \rho_t = 1$. Then we show the existence of a smooth Gauduchon factor on the smooth part of the central fibre.

Theorem 2.0.1 (cf. Corollary 2.2.4 and Theorem 2.3.1)**.** *In Setup (S), we have the following properties.*

(i) There is a uniform constant $C_G \geq 1$ such that the normalized Gauduchon factors ρ_t are bounded between 1 and C_G on each smooth fibre X_t for all $t \in \mathbb{D}^*_{1/2}$;

(ii) There exists a smooth bounded Gauduchon factor ρ of ω_0 on X_0^{reg}.

Thus, X_0 admits a *bounded Gauduchon metric* (see Definition 2.1.3) in the conformal class of ω_0. The idea of proof is to approximate the Gauduchon factor on the singular fibre by the normalized Gauduchon factors on nearby smooth fibres.

Next, we assume that X is a variety endowed with a bounded Gauduchon metric ω_{G}. We show that the trivial extension of the $(n-1, n-1)$-form $\omega_{\mathrm{G}}^{n-1}$ through X^{sing} is a pluriclosed current on X. Moreover, we also prove the analogous uniqueness result of Gauduchon.

Theorem 2.0.2. *Suppose that X is an n-dimensional, irreducible, reduced, compact complex space endowed with a bounded Gauduchon metric ω_{G}. Then we have the following uniqueness and extension properties.*

(i) If ω'_{G} is another bounded Gauduchon metric in the conformal class of ω_{G}, then ω'_{G} must be a positive multiple of ω_{G};

(ii) Let T be the positive $(n-1, n-1)$-current obtained as the trivial extension of $\omega_{\mathrm{G}}^{n-1}$. Then T is a pluriclosed current.

The main strategy is to use "good" cut-off functions. Complement of proper analytic subsets (eg. $X^{\mathrm{reg}} = X \setminus X^{\mathrm{sing}}$) admit exhaustion functions with small L^2-gradient. This enables us to show that the trivial extension of $\omega_{\mathrm{G}}^{n-1}$ as a current on X satisfies $\mathrm{dd^c} T = 0$ in the sense of currents globally on X. The uniqueness property follows similarly.

This note is organized as follows: Section 2.1 provides some backgrounds. Section 2.2 contains sup-estimate of normalized Gauduchon factors in families (the first property in Theorem 2.0.1). In Section 2.3, we show the existence of bounded Gauduchon factors on the central fibre (the second part of Theorem 2.0.1) and give the proof of Theorem 2.0.2.

2.1 Preliminaries

In this section, we recall some notation and conventions which will be used in the sequel. We define the twisted exterior derivative by $\mathrm{d^c} = \frac{\mathrm{i}}{2}(\bar\partial - \partial)$ and we then have $\mathrm{dd^c} = \mathrm{i}\partial\bar\partial$. We say that a form is *pluriclosed* if it is $\mathrm{dd^c}$-closed. We denote by:

- $\mathbb{D}_r := \{z \in \mathbb{C} \,|\, |z| < r\}$ the open disk of radius r;
- $\mathbb{D}_r^* := \{z \in \mathbb{C} \,|\, 0 < |z| < r\}$ the punctured disk of radius r.

When $r = 1$, we simply write $\mathbb{D} := \mathbb{D}_1$ and $\mathbb{D}^* := \mathbb{D}_1^*$.

2.1.1 Metrics on singular spaces

Let X be a reduced complex analytic space of pure dimension $n \geq 1$. We denote by X^{reg} the complex manifold of regular points of X and $X^{\mathrm{sing}} := X \setminus X^{\mathrm{reg}}$ the singular set of X. Now, we give the definition of hermitian metrics on reduced complex analytic space X.

Definition 2.1.1. A hermitian metric ω on X is the data of a hermitian metric ω on X^{reg} such that given any local embedding $X \underset{\text{loc.}}{\hookrightarrow} \mathbb{C}^N$, ω extends smoothly to a hermitian metric on $\mathbb{C}^N$.

Remark 2.1.2. The notion of smooth forms (and hermitian metrics) as above does not depend on the choice of local embeddings (see [Dem85, page 14]). Hermitian metrics always exist: one can use local embeddings and then glue local data of hermitian metrics by a partition of unity.

Note that in Definition 2.1.1 a hermitian metric on X is more than just a metric on X^{reg}. Now, we define Gauduchon metrics in the following two different concepts.

Definition 2.1.3. We say that a hermitian metric ω_{G} on X^{reg} is

(i) *Gauduchon* if it satisfies $\mathrm{dd^c}\omega_{\mathrm{G}}^{n-1} = 0$ on X^{reg};

(ii) *bounded Gauduchon metric* on X if there exist a hermitian metric ω and a positive bounded smooth function ρ defined on X^{reg} such that $\omega_{\mathrm{G}} = \rho^{\frac{1}{n-1}}\omega$ and $\mathrm{dd^c}\omega_{\mathrm{G}}^{n-1} = 0$ on X^{reg}.

We define the complex Laplacian and the norm of gradients with respect to ω by

$$\Delta_\omega f := \mathrm{tr}_\omega(\mathrm{dd^c} f) = \frac{n\mathrm{dd^c} f \wedge \omega^{n-1}}{\omega^n}$$
$$|\mathrm{d}f|^2_\omega := \mathrm{tr}_\omega(\mathrm{d}f \wedge \mathrm{d^c} f) = \frac{n\mathrm{d}f \wedge \mathrm{d^c} f \wedge \omega^{n-1}}{\omega^n}.$$

2.1.2 Currents on singular spaces

Recall that smooth forms on X are defined as restriction of smooth forms in local embeddings. We denote by

- $\mathcal{D}_{p,q}(X)$ the space of compactly supported smooth forms of bidegree (p,q);
- $\mathcal{D}_{p,p}(X)_{\mathbb{R}}$ the space of real smooth (p,p)-forms with compact support.

The notion of currents on X, $\mathcal{D}'_{p,q}(X)$ and $\mathcal{D}'_{p,p}(X)_{\mathbb{R}}$, is defined by acting on compactly supported smooth forms on X. The operators d, $\mathrm{d^c}$ and $\mathrm{dd^c}$ are well-defined by duality (see [Dem85] for detail arguments).

2.1.3 Example

We give an example of a non-Kähler variety satisfying Setup (S) extracted from [LT94]. The manifold $M = (\Gamma_1 \times \Gamma_2) \cap H \subset \mathbb{CP}^3 \times \mathbb{CP}^3$ is a simply connected Calabi–Yau threefold with $b_2 = 14$, where

$$\begin{aligned}
\Gamma_1 &= \{x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0\} \subset \mathbb{CP}^3,\\
\Gamma_2 &= \{y_0^3 + y_1^3 + y_2^3 + y_3^3 = 0\} \subset \mathbb{CP}^3,\\
H &= \{x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 = 0\} \subset \mathbb{CP}^3 \times \mathbb{CP}^3.
\end{aligned}$$

There exists 15 disjoint rational curves $\ell_1, \cdots, \ell_{15}$ with normal bundles $\mathcal{O}_{\mathbb{CP}^1}(-1)^{\oplus 2}$, $\ell_1, \cdots, \ell_{14}$ generate $H_2(M, \mathbb{Z})$, and there exists $a_j \neq 0$ for $j \in \{1, \cdots, 15\}$ such that $\sum_{j=1}^{15} a_j[\ell_j] = 0$. Then one can contract these curves and obtain a singular space X_0 with 15 ordinary double points. Moreover, there is a smoothing $\pi : \mathcal{X} \to \mathbb{D}$ of X_0 such that for all $t \neq 0$, X_t is diffeomorphic to a connected sum of $\mathbb{S}^3 \times \mathbb{S}^3$. This implies that X_t does not admit a Kähler metric since $b_2(X_t) = 0$. One can check that X_0 is not Kähler.

2.1.4 Remarks on the family setting

Under Setup (S), we have a smooth hermitian metric ω on $\mathcal{X}$ (as in Definition 2.1.1). For each $t \in \mathbb{D}$, it induces a smooth hermitain metric $\omega_t := \omega_{|X_t}$ on X_t. The family of metrics $(\omega_t)_t$ satisfies the following properties: For all $t \in \overline{\mathbb{D}}_{1/2}$, there is a constant $B \geq 0$ independent of t such that

$$-B\omega_t^n \leq \mathrm{dd}_t^c \omega_t^{n-1} \leq B\omega_t^n \tag{2.1.1}$$

where d_t^c is the twisted exterior derivative with respect to the complex structure of X_t. Indeed, in a local embedding, we have $-B\omega^n \leq \mathrm{dd^c}\omega^{n-1} \leq B\omega^n$ on $\mathcal{X}$; hence (2.1.1) is just the restriction on each fibre X_t. On the other hand, the volume of (X_t, ω_t) is comparable to the volume of (X_0, ω_0) for all $t \in \overline{\mathbb{D}}_{1/2}$. Namely, we have a uniform constant $C_V \geq 1$ such that

$$C_V^{-1} \leq \mathrm{Vol}_{\omega_t}(X_t) \leq C_V, \quad \forall t \in \overline{\mathbb{D}}_{1/2}. \tag{2.1.2}$$

The lower bound is obvious. One can prove the upper bound by the continuity of the total mass of currents $(\omega^n \wedge [X_t])_{t \in \overline{\mathbb{D}}_{1/2}}$. The proof goes as follows: the current of integration $[X_t]$ can be written as $\mathrm{dd^c} \log|\pi - t|$ by the Poincaré–Lelong formula. As $|\pi - t|$ converges to $|\pi|$

uniformly when $t \to 0$, $\log|\pi - t|$ converges to $\log|\pi|$ almost everywhere and, thus, $\log|\pi - t| \to \log|\pi|$ in L^1 when $t \to 0$ by Hartogs' lemma. Therefore, $\omega^n \wedge [X_t] \xrightarrow[t\to 0]{} \omega^n \wedge [X_0]$ in the sense of currents and this implies $\mathrm{Vol}_{\omega_t}(X_t) \xrightarrow[t\to 0]{} \mathrm{Vol}_{\omega_0}(X_0)$. Thus, using the compactness of X_0, we obtain a uniform upper bound C_V of $\mathrm{Vol}_{\omega_t}(X_t)$ for all $t \in \overline{\mathbb{D}}_{1/2}$.

Finally, we give a remark on non-smoothable singularities: The first non-smoothable example was given by Thom and reproduced by Hartshorne [Har74]. They considered a cone in $\mathbb{C}^6$ over the Segre embedding of $\mathbb{CP}^1 \times \mathbb{CP}^2$ into $\mathbb{CP}^5$ and they proved that the cone does not admit a smoothing because of a topological constraint.

2.2 Gauduchon metrics on smooth fibres

The aim of this section is to prove the uniform boundedness of normalized Gauduchon factors ρ_t with respect to ω_t on smooth fibres X_t in Setup (S). First, we fix a compact hermitian manifold (X, ω). Suppose that $B \geq 0$ is a constant such that $-B\omega^n \leq dd^c\omega^{n-1} \leq B\omega^n$. From Gauduchon's theorem [Gau77], there exists a unique positive smooth function $\rho \in \ker \Delta_\omega^*$ or, equivalently

$$dd^c(\rho\omega^{n-1}) = dd^c\rho \wedge \omega^{n-1} + d\rho \wedge d^c\omega^{n-1} - d^c\rho \wedge d\omega^{n-1} + \rho dd^c\omega^{n-1} = 0$$

such that $\inf_X \rho = 1$ and $\omega_G = \rho^{\frac{1}{n-1}}\omega$ is a Gauduchon metric. Then we prove that the Gauduchon factor is bounded by geometric quantities.

Theorem 2.2.1. *Suppose that (X, ω) is an n-dimensional compact hermitian manifold. If $\rho \in \ker \Delta_\omega^*$ and $\inf_X \rho = 1$, then there is a constant C_G depending only on n, B, C_S, C_P and $\mathrm{Vol}_\omega(X)$ such that*

$$\sup_X \rho \leq C_G,$$

where C_S and C_P are Sobolev and Poincaré constants.

The proof of Theorem 2.2.1 is inspired by the paper of Tosatti and Weinkove [TW10]. We apply Moser's iteration twice to obtain an upper bound of ρ. On the other hand, under Setup (S), the Sobolev and Poincaré constants of the fibres X_t are uniformly bounded independently of t. The uniform Sobolev constant in family comes from Wirtinger inequality and Michael and Simon's Sobolev inequality on minimal submanifolds [MS73]. The study of Poincaré constant in families goes back to Yoshikawa [Yos97] and Ruan and Zhang [RZ11]. For convenience, the reader is also referred to [DGG20, Proposition 3.8 and 3.10]. Although they only stated the properties on a family of Kähler spaces, the proof does not rely on Kähler structures.

Proposition 2.2.2. *Suppose that $\pi : (\mathcal{X}, \omega) \to \mathbb{D}$ is a family of compact hermitian varieties in Setup (S). For all $t \in K \Subset \mathbb{D}$, there exists uniform Sobolev and Poincaré constants $C_S(K)$ and $C_P(K)$*

such that

$$\forall f \in L_1^2(X_t^{\text{reg}}), \quad \left(\int_{X_t} |f|^{\frac{2n}{n-1}} \omega_t^n \right)^{\frac{n-1}{n}} \le C_S \left(\int_{X_t} |df|^2_{\omega_t} \omega_t^n + \int_{X_t} |f|^2 \omega_t^n \right)$$

and

$$\forall f \in L_1^2(X_t^{\text{reg}}) \text{ and } \int_{X_t} f \omega_t^n = 0, \quad \int_{X_t} |f|^2 \omega_t^n \le C_P \int_{X_t} |df|^2_{\omega_t} \omega_t^n.$$

Remark 2.2.3. The irreducible condition is crucial in the proof of uniform Poincaré inequality. Assume that X_0 has two irreducible components X_0' and X_0''. Consider a function f on X_0^{reg} defined by

$$\begin{cases} f = 1/\operatorname{Vol}_{\omega_t}(X_0') & \text{on } (X_0')^{\text{reg}} \\ f = -1/\operatorname{Vol}_{\omega_t}(X_0'') & \text{on } (X_0'')^{\text{reg}} \\ f = 0 & \text{otherwise} \end{cases}.$$

Then it is not hard to see that the RHS of Poincaré inequality is zero but the LHS is positive. This yields a contradiction. One can also construct a "quantitative" version of that example. Namely, the Poincaré constant $C_{P,t}$ on each smooth fibre X_t blows up when $t \to 0$ (see, eg., [Yos97, DGG20]).

Combining Theorem 2.2.1, Proposition 2.2.2, (2.1.1) and (2.1.2), we obtain the following uniform estimate in the family setting.

Corollary 2.2.4. *Let $\pi : (\mathcal{X}, \omega) \to \mathbb{D}$ be a family of compact hermitian manifolds as in Setup (S). Then there exists a constant $C_G > 0$ such that*

$$\sup_{X_t} \rho_t \le C_G, \text{ and } \inf_{X_t} \rho_t = 1 \quad \text{for all } t \in \mathbb{D}^*_{1/2}$$

where ρ_t is the Gauduchon factor with respect to (X_t, ω_t).

2.2.1 Proof of Theorem 2.2.1

In this subsection, we establish two gradient estimates ((2.2.1) and (2.2.3)) and then apply Moser's iteration argument to obtain an upper bound of ρ. In order to check the dependence on each data, we formulate the dependence of given constants. For convenience, we write $V := \operatorname{Vol}_\omega(X)$. We start the proof with a useful formula.

Lemma 2.2.5. *Suppose that $\rho \in \ker \Delta^*_\omega$. Then we have*

$$\int_X F'(\rho) d\rho \wedge d^c\rho \wedge \omega^{n-1} = \int_X G(\rho) dd^c \omega^{n-1}$$

where F and G are two real $\mathcal{C}^1$-functions defined on $\mathbb{R}_{>0}$ which satisfy $xF'(x) = G'(x)$.

Proof. As $\rho \in \ker \Delta_\omega^*$, we have $\mathrm{dd^c}(\rho\omega^{n-1}) = 0$. From Stokes' theorem, it follows that

$$\begin{aligned}0 &= \int_X F(\rho)\mathrm{dd^c}(\rho\omega^{n-1}) = -\int_X F'(\rho)\mathrm{d}\rho \wedge \mathrm{d^c}(\rho\omega^{n-1}) \\ &= -\int_X F'(\rho)\mathrm{d}\rho \wedge \mathrm{d^c}\rho \wedge \omega^{n-1} - \int_X F'(\rho)\rho\mathrm{d}\rho \wedge \mathrm{d^c}\omega^{n-1}.\end{aligned}$$

By the assumption $xF'(x) = G'(x)$, we obtain the desired formula

$$\begin{aligned}\int_X F'(\rho)\mathrm{d}\rho \wedge \mathrm{d^c}\rho \wedge \omega^{n-1} &= -\int_X \rho F'(\rho)\mathrm{d}\rho \wedge \mathrm{d^c}\omega^{n-1} \\ &= -\int_X \mathrm{d}G(\rho) \wedge \mathrm{d^c}\omega^{n-1} = \int_X G(\rho)\mathrm{dd^c}\omega^{n-1}.\end{aligned}$$

□

Consider $F'(x) = \frac{np^2}{4}x^{-p-2}$ and $G(x) = -\frac{np}{4}x^{-p}$ for $p \geq 1$. By Lemma 2.2.5, we have the following inequality

$$\begin{aligned}\int_X \left|\mathrm{d}\left(\rho^{-\frac{p}{2}}\right)\right|_\omega^2 \omega^n &= \int_X n\mathrm{d}\left(\rho^{-\frac{p}{2}}\right) \wedge \mathrm{d^c}\left(\rho^{-\frac{p}{2}}\right) \wedge \omega^{n-1} \\ &= \int_X F'(\rho)\mathrm{d}\rho \wedge \mathrm{d^c}\rho \wedge \omega^{n-1} = \int_X G(\rho)\mathrm{dd^c}\omega^{n-1} \\ &= \frac{np}{4}\int_X \rho^{-p}\left(-\mathrm{dd^c}\omega^{n-1}\right) \\ &\leq p\frac{nB}{4}\int_X \rho^{-p}\omega^n.\end{aligned} \tag{2.2.1}$$

Put $\beta = \frac{n}{n-1} > 1$. Combining Sobolev inequality and (2.2.1), we obtain

$$\begin{aligned}\left(\int_X (\rho^{-1})^{p\beta}\omega^n\right)^{\frac{1}{\beta}} &\leq C_S\left(\int_X \left|\mathrm{d}\left(\rho^{-\frac{p}{2}}\right)\right|_\omega^2 \omega^n + \int_X \left|\rho^{-\frac{p}{2}}\right|^2 \omega^n\right) \\ &\leq C_S\left(p\frac{nB}{4}\int_X (\rho^{-1})^p\omega^n + \int_X (\rho^{-1})^p\omega^n\right) \\ &\leq p\left(\frac{nB}{4} + 1\right)C_S\int_X \rho^{-p}\omega^n.\end{aligned}$$

For all $p \geq 1$, we have

$$\left\|\rho^{-1}\right\|_{L^{p\beta}(X,\omega)} \leq p^{\frac{1}{p}}C_1^{\frac{1}{p}}\left\|\rho^{-1}\right\|_{L^p(X,\omega)}$$

where C_1 is a constant depending only on n, B, C_S. Inductively, we obtain

$$\begin{aligned}\left\|\rho^{-1}\right\|_{L^{p\beta^k}(X,\omega)} &\le p^{\frac{1}{p}\sum_{j=0}^{k-1}\frac{1}{\beta^j}}\beta^{\frac{1}{p}\sum_{j=0}^{k-1}\frac{j}{\beta^j}}C_1^{\frac{1}{p}\sum_{j=0}^{k-1}\frac{1}{\beta^j}}\left\|\rho^{-1}\right\|_{L^p(X,\omega)}\\ &\le p^{\frac{n}{p}}\beta^{\frac{n(n-1)}{p}}C_1^{\frac{n}{p}}\left\|\rho^{-1}\right\|_{L^p(X,\omega)}.\end{aligned}$$

Let $p = 1$ and $k \to \infty$ and, thus, we obtain

$$1 = \sup_X \rho^{-1} \le \beta^{n(n-1)}C_1^n \int_X \rho^{-1}\omega^n.$$

Therefore, the L^1-norm of ρ^{-1} is bounded away from zero by a constant δ which depends only on n, B, C_S:

$$\int_X \rho^{-1}\omega^n \ge \frac{1}{\beta^{n(n-1)}C_1^n} =: 2\delta.$$

Choosing sufficiently small δ, we may assume $A := V/\delta \ge 1$. Then we have

$$2\delta \le \int_{\{\rho<A\}}\rho^{-1}\omega^n + \int_{\{\rho\ge A\}}\rho^{-1}\omega^n \le \int_{\{\rho<A\}}\omega^n + \frac{1}{A}\int_X \omega^n.$$

Hence, the volume of $\{\rho < A\}$ is bounded away from zero:

$$\int_{\{\rho<A\}}\omega^n \ge \delta. \tag{2.2.2}$$

Now, we consider $F'(x) = \frac{n(p+1)^2}{4}\frac{(\log x)^{p-1}}{x^2}$ and $G(x) = \frac{n(p+1)^2}{4p}(\log x)^p$ for $p \ge 1$. From Lemma 2.2.5, we find the following estimate

$$\begin{aligned}\int_X \left|d(\log\rho)^{\frac{p+1}{2}}\right|^2_\omega \omega^n &= \frac{n(p+1)^2}{4}\int_X \frac{(\log\rho)^{p-1}}{\rho^2}d\rho\wedge d^c\rho\wedge\omega^{n-1}\\ &= \int_X F'(\rho)d\rho\wedge d^c\rho\wedge\omega^{n-1} = \int_X G(\rho)dd^c\omega^{n-1}\\ &= \frac{n(p+1)^2}{4p}\int_X(\log\rho)^p dd^c\omega^{n-1}\\ &\le pnB\int_X(\log\rho)^p\omega^n.\end{aligned} \tag{2.2.3}$$

Again, Sobolev inequality, (2.2.3) and Hölder inequality yield the following inequalities:

$$\begin{aligned}\left(\int_X (\log\rho)^{(p+1)\beta}\omega^n\right)^{\frac{1}{\beta}} &\leq C_S\left(\int_X \left|\mathrm{d}(\log\rho)^{\frac{p+1}{2}}\right|^2_\omega \omega^n + \int_X (\log\rho)^{p+1}\omega^n\right)\\ &\leq C_S\left(pnB\int_X (\log\rho)^p\omega^n + \int_X (\log\rho)^{p+1}\omega^n\right)\\ &\leq C_S\left(pnBV^{\frac{1}{p+1}}\left(\int_X (\log\rho)^{p+1}\omega^n\right)^{\frac{p}{p+1}} + \int_X (\log\rho)^{p+1}\omega^n\right)\\ &\leq (p+1)2nBC_S\max\{V,1\}\max\left\{\int_X (\log\rho)^{p+1}\omega^n, 1\right\}.\end{aligned}$$

Write $q = p+1 \geq 2$. We have

$$\|\log\rho\|_{L^{q\beta}(X,\omega)} \leq q^{\frac{1}{q}} C_2^{\frac{1}{q}} \max\left\{\|\log\rho\|_{L^q(X,\omega)}, 1\right\}$$

where $C_2 > 0$ is a constant depending only on n, B, C_S, V. Using the similar strategy of Moser's iteration again, we derive

$$\begin{aligned}\sup_X(\log\rho) &\leq 2^{\frac{n}{2}}\beta^{\frac{n(n-1)}{2}} C_2^{\frac{n}{2}} \max\{\|\log\rho\|_{L^2(X,\omega)}, 1\}\\ &\leq 2^{\frac{n}{2}}\beta^{\frac{n(n-1)}{2}} C_2^{\frac{n}{2}} \max\left\{(\sup_X \log\rho)^{\frac{1}{2}}\left(\int_X \log\rho\,\omega^n\right)^{\frac{1}{2}}, 1\right\}\end{aligned}$$

and, thus,

$$\sup_X(\log\rho) \leq C_3 \max\left\{\int_X \log\rho\,\omega^n, 1\right\}$$

for some constant $C_3 = C_3(n, B, C_S, V)$.

Now, everything comes down to bounding $\int_X \log\rho\,\omega^n$ from above. Using Poincaré inequality and (2.2.3) with $p = 1$, we obtain

$$\int_X \left|\log\rho - \underline{\log\rho}\right|^2 \omega^n \leq C_P \int_X |\mathrm{d}\log\rho|^2_\omega\,\omega^n \leq C_4 \int_X \log\rho\,\omega^n \tag{2.2.4}$$

where $\underline{\log\rho} = \frac{1}{V}\int_X \log\rho\,\omega^n$ is the average of $\log\rho$ and $C_4 = C_4(n, B, C_P)$. Then by (2.2.2), we can infer that

$$\begin{aligned}\delta\int_X \log\rho\,\omega^n = V\delta\underline{\log\rho} &\leq V\int_{\{\rho<A\}} \underline{\log\rho\,\omega^n}\\ &\leq V\int_{\{\rho<A\}} \left(\log A - \log\rho + \underline{\log\rho}\right)\omega^n\\ &\leq V\int_X \left(\left|\log\rho - \underline{\log\rho}\right| + \log A\right)\omega^n.\end{aligned} \tag{2.2.5}$$

We use (2.2.4) and (2.2.5) to obtain

$$\begin{aligned}\int_X \log\rho\omega^n &\le \frac{V}{\delta}\left(\int_X \left|\log\rho - \underline{\log\rho}\right|\omega^n + V\log A\right)\\ &\le \frac{V}{\delta}\left(V^{\frac{1}{2}}\left(\int_X \left|\log\rho - \underline{\log\rho}\right|^2\omega^n\right)^{\frac{1}{2}} + V\log A\right)\\ &\le \frac{V}{\delta}\left(V^{\frac{1}{2}}C_4^{\frac{1}{2}}\left(\int_X \log\rho\omega^n\right)^{\frac{1}{2}} + V\log A\right).\end{aligned}$$

Note that if $x^2 \le ax + b$ for $a, b > 0$, then $x \le \frac{a}{2} + (b + \frac{a^2}{4})^{\frac{1}{2}}$. Eventually, we obtain

$$\int_X \log\rho\omega^n \le C_5(n, B, C_S, C_P, V)$$

and this completes the proof of Theorem 2.2.1.

2.3 Gauduchon current on the singular fibre

In this section, we construct a Gauduchon factor ρ_0 on X_0^{reg} as the limit of the Gauduchon factors on the nearby fibres X_t. In particular, we derive that the limit ρ is bounded and, thus, $\rho^{\frac{1}{n-1}}\omega_0$ is a bounded Gauduchon metric on X_0^{reg}. On the other hand, for a fixed irreducible, reduced, compact complex analytic space X, we show that the $(n-1)$-power of a bounded Gauduchon metric can be extend trivially to a pluriclosed current on whole X and also prove a uniqueness result.

2.3.1 Gauduchon metric on the central fibre

Theorem 2.3.1. *Suppose that X_0 is the central fibre in Setup (S). There exists a smooth function $1 \le \rho \le C_G$ on X_0^{reg} such that $\rho^{\frac{1}{n-1}}\omega_0$ is a bounded Gauduchon metric. Here C_G is the constant introduced in Corollary 2.2.4.*

Proof. We shall apply standard elliptic theory on some relatively compact subsets of X_0^{reg} to get a smooth function ρ. This ρ is the limit of $(\rho_{t_j})_{j\in\mathbb{N}}$ defined on the fibre X_{t_j} for some sequence $t_j \to 0$ when $j \to +\infty$. In the sequel, we shall denote by $U \Subset V$ if the closure of U is a compact subset in V.

Recall that $\pi : \mathcal{X} \to \mathbb{D}$ is submersive on X_0^{reg}. By the tubular neighborhood theorem, there is an open neighborhood $\mathcal{M}$ of X_0^{reg} in $\mathcal{X}^{\text{reg}}$ such that the following statements hold

(i) For all $U \Subset X_0^{\text{reg}}$, there exists an open set $\mathcal{M}_U \subset \mathcal{X}^{\text{reg}}$, a constant $\delta_U > 0$, and a diffeomorphism $\psi_U : U \times \mathbb{D}_{\delta_U} \xrightarrow{\sim} \mathcal{M}_U$ such that the diagram

$$\begin{array}{ccc} U \times \mathbb{D}_{\delta_U} & \xrightarrow[\text{diffeo.}]{\psi_U} & \mathcal{M}_U \subset \mathcal{M} \subset \mathcal{X}^{\mathrm{reg}} \\ \big\downarrow \mathrm{pr}_2 & \swarrow \pi & \\ \mathbb{D}_{\delta_U} & & \end{array}$$

commutes. In particular, for all $t \in \mathbb{D}_{\delta_U}$, $\psi_U(\cdot, t) : U \to \mathcal{M}_U$ is a diffeomorphism onto its image $M_{U,t} = \psi_U(U, t) = \mathcal{M}_U \cap X_t$.

(ii) If $U \Subset V \Subset X_0^{\mathrm{reg}}$, we have $\delta_U \geq \delta_V > 0$ and $\psi_U(x, t) = \psi_V(x, t)$ for all $x \in U$ and $t \in \mathbb{D}_{\delta_V}$.

Now, we denote by $P_t = \Delta^*_{\omega_t}$ and fix $U_1 \Subset U_2 \Subset X_0^{\mathrm{reg}}$ which are connected open subsets. Note that U_i can be identified with $M_{U_i,t} = \psi_{U_2}(U_i, t)$ for $i \in \{1, 2\}$ and for all $t \in \mathbb{D}_{\delta_{U_2}}$ and, hence, P_t can act on smooth functions defined on U_2. In addition, on U_2, the Riemannian metric g_0 induced by ω_0 is quasi-isometric to g_t induced by ω_t, and the volume form ω_0^n is comparable with ω_t^n. In other words, we have a uniform constant $C_{U_2} > 0$ such that

$$C_{U_2}^{-1} \langle \cdot, \cdot \rangle_{\omega_0} \leq \langle \cdot, \cdot \rangle_{\omega_t} \leq C_{U_2} \langle \cdot, \cdot \rangle_{\omega_0}, \text{ and } C_{U_2}^{-1} \omega_0^n \leq \omega_t^n \leq C_{U_2} \omega_0^n. \tag{2.3.1}$$

By Gårding inequality, we have

$$\|u\|_{L^2_2(U_1,\omega_0)} \leq C_{U_1,U_2} \left(\|P_t u\|_{L^2(U_2,\omega_0)} + \|u\|_{L^2(U_2,\omega_0)} \right)$$

for all $u \in \mathcal{C}_c^\infty(U_2)$. The constant C_{U_1,U_2} can be chosen independent of t because the coefficients of P_t move smoothly in t. Choose a cut-off function χ such that $\mathrm{supp}(\chi) \subset U_2$ and $\chi \equiv 1$ on U_1. From Gårding inequality, we obtain

$$\|\rho_t\|_{L^2_2(U_1,\omega_0)} \leq C_{U_1,U_2} \left(\|P_t(\chi\rho_t)\|_{L^2(U_2,\omega_0)} + \|\chi\rho_t\|_{L^2(U_2,\omega_0)} \right). \tag{2.3.2}$$

In (2.3.2), the second term $\|\chi\rho_t\|_{L^2(U_2,\omega_0)}$ is uniformly bounded because of Corollary 2.2.4. Hence, we only need to estimate $\|P_t(\chi\rho_t)\|_{L^2(U_2,\omega_0)}$. Note that

$$\begin{aligned} P_t(\chi\rho_t) &= \frac{n}{\omega_t^n} \mathrm{dd}^c_t(\chi\rho_t\omega_t^{n-1}) \\ &= \frac{n}{\omega_t^n} \left(2\mathrm{d}\rho_t \wedge \mathrm{d}^c_t\chi \wedge \omega_t^{n-1} + \chi\mathrm{dd}^c_t(\rho_t\omega_t^{n-1}) + \rho_t\mathrm{dd}^c(\chi\omega_t^{n-1}) - \chi\rho_t\mathrm{dd}^c_t\omega_t^{n-1} \right) \\ &= 2\langle \mathrm{d}\rho_t, \mathrm{d}\chi \rangle_{\omega_t} + \rho_t P_t(\chi) - \rho_t\chi \frac{\mathrm{dd}^c_t\omega_t^n}{\omega_t^n}. \end{aligned} \tag{2.3.3}$$

Obviously, $\rho_t P_t(\chi)$ and $\rho_t\chi\frac{\mathrm{dd}^c_t\omega_t^{n-1}}{\omega_t^n}$ are uniformly bounded, so we only need to control the L^2-

norm of the first term $\langle d\rho_t, d\chi\rangle_{\omega_t}$:

$$\begin{aligned}\int_{U_2} \left|\langle d\rho_t, d\chi\rangle_{\omega_t}\right|^2 \omega_0^n &\leq C_{U_2}^2 \left(\sup_{U_2} |d\chi|^2_{\omega_0}\right) \int_{U_2} |d\rho_t|^2_{\omega_t} \omega_t^n \\ &= C_{U_2}^2 \left(\sup_{U_2} |d\chi|^2_{\omega_0}\right) \int_{M_{U_2,t}} |d\rho_t|^2_{\omega_t} \omega_t^n \\ &\leq C_{U_2}^2 \left(\sup_{U_2} |d\chi|^2_{\omega_0}\right) \int_{X_t} |d\rho_t|^2_{\omega_t} \omega_t^n \\ &\leq C_{U_2}^2 \left(\sup_{U_2} |d\chi|^2_{\omega_0}\right) \frac{nB}{2} \int_{X_t} \rho_t^2 \omega_t^n.\end{aligned}$$

Here the first line is by Cauchy–Schwarz inequality and (2.3.1). The fourth line follows from an argument similar to that used in (2.2.1) (just replace $-p$ by p). As $1 \leq \rho_t \leq C_G$ and $\mathrm{Vol}_{\omega_t}(X_t) \leq C_V$, we find a uniform bound of $\|P_t(\chi\rho_t)\|_{L^2(U_2,\omega_0)}$. Hence, $\|\rho_t\|_{L^2_2(U_1,\omega_0)}$ is uniformly bounded by some uniform constant $C(U_1, U_2)$.

For higher-order estimates, we apply higher-order Gårding inequalities on the fixed domains $U_1 \Subset U_2 \Subset X_0^{\mathrm{reg}}$:

$$\|u\|_{L^2_{s+2}(U_1,\omega_0)} \leq C_{s,U_1,U_2} \left(\|P_t u\|_{L^2_s(U_2,\omega_0)} + \|u\|_{L^2(U_2,\omega_0)}\right)$$

for all $u \in \mathcal{C}_c^\infty(U_2)$. Let $\mathcal{U} = (U_i)_{i\in\mathbb{N}}$ be a relatively compact exhaustion of X_0^{reg}. Differentiating (2.3.3) on both sides and using a bootstrapping argument, we obtain $\|\rho_t\|_{L^2_s(U_1,\omega_0)} < C(s,\mathcal{U})$ where $C(s,\mathcal{U})$ does not depend on t. By Rellich's theorem, there exists a subsequence $(\rho_{t_j})_{j\in\mathbb{N}}$ such that ρ_{t_j} converges to ρ in $\mathcal{C}^k(\overline{U_1})$ for all $k \in \mathbb{N}$ when $t_j \to 0$. Therefore $dd_0^c(\rho\omega_0^{n-1}) = \lim_{j\to+\infty} dd_{t_j}^c(\rho_{t_j}\omega_{t_j}^{n-1}) = 0$ on U_1. Using a diagonal argument, we can infer that there is a smooth function ρ on X_0^{reg} which is bounded between 1 and C_G, and satisfies $dd_0^c(\rho\omega_0^{n-1}) = 0$ on X_0^{reg}. □

2.3.2 Proof of Theorem 2.0.2

In this subsection, we always assume that X is an irreducible reduced compact complex space and ω_G is a bounded Gauduchon metric on X. Before proving the uniqueness result and extension property, we recall the existence of cut-off functions with small L^2-gradients. From a classical property in Riemannian geometry, these cut-off functions do exist on so-called *parabolic* manifolds (see eg. [Gla83, EG92] and references therein). A Riemannian manifold (M, g) is said to be parabolic if for all compact $K \subset M$ and each $\varepsilon > 0$, there exists a smooth cut-off function $\chi \in \mathcal{C}_c^\infty(M)$ with $0 \leq \chi \leq 1$, $\chi \equiv 1$ on a neighborhood of K and $\|d\chi\|_{L^2(M,g)} \leq \varepsilon$. In our case, one can easily construct explicit cut-off functions by Hironaka's desingularization and log log-potentials (cf. [Ber12, Lemma 2.2] and [CGP13, Section 9]):

Lemma 2.3.2. *Suppose that (X,ω) is a compact hermitian variety. There exist cut-off functions $(\chi_\varepsilon)_{\varepsilon>0} \subset \mathcal{C}^\infty_c(X^{\mathrm{reg}})$ satisfying the following properties:*

(i) χ_ε *is increasing to the characteristic function of* X^{reg} *when* ε *decreases to zero;*

(ii) $\int_X \left|dd^c\chi_\varepsilon \wedge \omega^{n-1}\right| \to 0$ *when* $\varepsilon \to 0$;

(iii) $\int_X d\chi_\varepsilon \wedge d^c\chi_\varepsilon \wedge \omega^{n-1} \to 0$ *when* $\varepsilon \to 0$.

These functions allow us to perform integration by parts as on a compact manifold. Then we argue that the desired results hold when ε tends to zero. Now, we give a proof of Theorem 2.0.2:

Proof of Theorem 2.0.2. We divide the proof in two parts.

Part 1: Uniqueness of singular Gauduchon metrics. Assume that ω_G and ω'_G are both bounded Gauduchon metrics in the same conformal class. We write ρ to be the bounded Gauduchon factor satisfying $\rho^{\frac{1}{n-1}}\omega_G = \omega'_G$. Let $(\chi_\varepsilon)_{\varepsilon>0}$ be cut-off functions given in Lemma 2.3.2. From Stokes formula and direct computations, we derive

$$\int_{X^{\mathrm{reg}}} \chi_\varepsilon |d\rho|^2_{\omega_G}\omega_G^n = \frac{1}{2}\int_{X^{\mathrm{reg}}} \rho^2 dd^c\chi_\varepsilon \wedge \omega_G^{n-1}. \tag{2.3.4}$$

Indeed,

$$\begin{aligned}
\int_{X^{\mathrm{reg}}} \chi_\varepsilon |d\rho|^2_{\omega_G}\omega_G^{n-1} &= \int_{X^{\mathrm{reg}}} \chi_\varepsilon d\rho \wedge d^c\rho \wedge \omega_G^{n-1} = -\int_{X^{\mathrm{reg}}} \rho d\left(\chi_\varepsilon d^c\rho \wedge \omega_G^{n-1}\right)\\
&= -\int_{X^{\mathrm{reg}}} \rho\left(d\chi_\varepsilon \wedge d^c\rho \wedge \omega_G^{n-1} + \chi_\varepsilon dd^c\rho \wedge \omega_G^{n-1} - \chi_\varepsilon d^c\rho \wedge d\omega_G^{n-1}\right)\\
&= -\frac{1}{2}\int_{X^{\mathrm{reg}}} d\rho^2 \wedge d^c\chi_\varepsilon \wedge \omega_G^{n-1} + \int_{X^{\mathrm{reg}}} \rho\chi_\varepsilon d\rho \wedge d^c\omega_G^{n-1}\\
&= \frac{1}{2}\int_{X^{\mathrm{reg}}} \rho^2 d\left(d^c\chi_\varepsilon \wedge \omega_G^{n-1}\right) - \frac{1}{2}\int_{X^{\mathrm{reg}}} \rho^2 d\left(\chi_\varepsilon d^c\omega_G^{n-1}\right)\\
&= \frac{1}{2}\int_{X^{\mathrm{reg}}} \rho^2 dd^c\chi_\varepsilon \wedge \omega_G^{n-1} - \frac{1}{2}\int_{X^{\mathrm{reg}}} \rho^2 d^c\chi_\varepsilon \wedge d\omega_G^{n-1} - \frac{1}{2}\int_{X^{\mathrm{reg}}} \rho^2 d\chi_\varepsilon \wedge d^c\omega_G^{n-1}\\
&= \frac{1}{2}\int_{X^{\mathrm{reg}}} \rho^2 dd^c\chi_\varepsilon \wedge \omega_G^{n-1}.
\end{aligned}$$

By Lemma 2.3.2 and the boundedness of ω_G, we can see that $\int_X |dd^c\chi_\varepsilon \wedge \omega_G^{n-1}|$ converges to zero when ε tends to zero. As ρ is bounded, the RHS of (2.3.4) goes to zero and, hence,

$$\int_{X^{\mathrm{reg}}} |d\rho|^2_{\omega_G}\omega_G^n = 0.$$

Because X^{reg} is connected by the irreducible assumption, ρ is a constant.

Part 2: Extension to a pluriclosed current. We already have a smooth pluriclosed bounded positive $(n-1,n-1)$-form ω_G^{n-1} on X^{reg}. Note that ω_G^{n-1} can extend trivially as a bounded positive $(n-1,n-1)$-current T on X. If ω_G^{n-1} is closed, so would be T by the Skoda–El Mir extension result [Sko82, El 84]. Some results for plurisubharmonic currents do exist in the literature (see, eg., [AB93, DEE03]), but we could not find the one of interest for us here. Therefore, we provide a quick proof.

Again, we are going to use good cut-off functions $(\chi_\varepsilon)_{\varepsilon>0}$ in Lemma 2.3.2 to prove that T is pluriclosed in the sense of currents. Fix a smooth function f on X. We need to show that

$$\langle f, \mathrm{dd^c} T\rangle := \int_X \mathrm{dd^c} f \wedge T = 0.$$

Using the cut-off functions constructed in Lemma 2.3.2, we can write

$$\int_X \mathrm{dd^c} f \wedge T = \underbrace{\int_{R_\varepsilon} \chi_\varepsilon \mathrm{dd^c} f \wedge T}_{:=\mathrm{I}_\varepsilon} + \underbrace{\int_{S_\varepsilon} (1-\chi_\varepsilon)\mathrm{dd^c} f \wedge T}_{:=\mathrm{II}_\varepsilon}$$

where R_ε is a small open neighborhood of the closure of $\{\chi_\varepsilon > 0\}$ contained in X^{reg} and similarly S_ε is a small neighborhood of the closure of $\{\chi_\varepsilon < 1\}$. According to Lemma 2.3.2, as $\varepsilon \to 0$, R_ε tends to X^{reg} and S_ε shrinks to X^{sing}. Therefore, II_ε converges to zero when ε goes to zero. We compute the term I_ε by the Stokes formula:

$$\begin{aligned}\mathrm{I}_\varepsilon &= \int_{R_\varepsilon} f \mathrm{dd^c}\left(\chi_\varepsilon \omega_{\mathrm{G}}^{n-1}\right) = \int_{R_\varepsilon} f\left(\mathrm{dd^c}\chi_\varepsilon \wedge \omega_{\mathrm{G}}^{n-1} + 2\mathrm{d}\omega_{\mathrm{G}}^{n-1} \wedge \mathrm{d^c}\chi_\varepsilon\right) \\ &= -\underbrace{\int_{R_\varepsilon} f \mathrm{dd^c}\chi_\varepsilon \wedge \omega_{\mathrm{G}}^{n-1}}_{:=\mathrm{III}_\varepsilon} \underbrace{-2\int_{R_\varepsilon} \mathrm{d}f \wedge \mathrm{d^c}\chi_\varepsilon \wedge \omega_{\mathrm{G}}^{n-1}}_{:=\mathrm{IV}_\varepsilon}.\end{aligned}$$

By Cauchy–Schwarz inequality, the term IV_ε can be bounded by

$$|\mathrm{IV}_\varepsilon| \leq \underbrace{\int_{R_\varepsilon} \mathbb{1}_{\{\mathrm{d}\chi_\varepsilon \neq 0\}} \mathrm{d}f \wedge \mathrm{d^c} f \wedge \omega_{\mathrm{G}}^{n-1}}_{:=\mathrm{V}_\varepsilon} + \underbrace{\int_{R_\varepsilon} \mathrm{d}\chi_\varepsilon \wedge \mathrm{d^c}\chi_\varepsilon \wedge \omega_{\mathrm{G}}^{n-1}}_{:=\mathrm{VI}_\varepsilon}.$$

Using the dominated convergence theorem, V_ε converges to zero when ε goes to zero, because the set $\{\mathrm{d}\chi_\varepsilon \neq 0\}$ is contained in S_ε and S_ε shrinks to X^{sing}. Applying Lemma 2.3.2, III_ε and VI_ε converge to zero as ε tending to zero. These yield that $\mathrm{I}_\varepsilon \to 0$ as $\varepsilon \to 0$. All in all, we have $\int_X \mathrm{dd^c} f \wedge T = 0$ for all $f \in \mathcal{C}^\infty(X)$; hence, $\mathrm{dd^c} T = 0$ in the sense of currents. □

Chapter 3

Singular Gauduchon metrics, again

We prove a singular version of Gauduchon's theorem on arbitrary compact hermitian varieties by assuming the existence of a uniform Sobolev constant. In particular, we prove a uniform estimate of normalized Gauduchon factors in families of hermitian varieties. Then we use singular Gauduchon metrics to establish a Sup-L^1 comparison of quasi-plurisubharmonic functions in families of hermitian varieties where all fibres have only isolated singularities.

Introduction

Gauduchon metrics, first introduced by Gauduchon [Gau77], is a generalization of Kähler metrics on non-Kähler manifolds. It is a natural class of hermitian metrics which exist on all compact complex manifolds. A hermitian metric ω_{G} on an n-dimensional compact complex manifold X is called Gauduchon if $dd^c\omega_{\mathrm{G}}^{n-1} = 0$, and the classical result of Gauduchon says that every hermitian metrics is conformally equivalent to a Gauduchon metric (uniquely up to scaling when $n \geq 2$).

Constructing canonical or special metrics on complex manifolds is a central subject in complex geometry. In Kähler geometry, Yau's celebrated solution of the Calabi conjecture [Yau78] and Uhlenbeck–Yau's characterization of the existence of hermitian–Einstein metrics on stable bundles [UY86] are landmarks in canonical metrics problems. Much interest has been devoted to study non-Käher Calabi–Yau manifolds, following "Reid's fantasy" [Rei87] that all Calabi–Yau threefolds should form connected families provided one allows deformations and singular transitions through non-Kähler Calabi–Yau manifolds. To understand non-Kähler objects, one may ask how much metrical theories carry over from the Kähler to non-Kähler geometry. In these topics, Gauduchon metrics play an important role. For instance, using Gauduchon metrics, Li and Yau [LY87] defined the notion of slope stability on non-Kähler manifolds and then they generalized Uhlenbeck–Yau's theorem in non-Kähler context. Motivated by Yau's theorem, Gauduchon [Gau84, IV.5] posed a Calabi–Yau type question about Gauduchon metrics, which has been solved by Székelyhidi, Tosatti, and Weinkove [STW17]. In a different direction, Angella–Calamai–Spotti [ACS17] have studied the Chern–Yamabe problem. They use Gauduchon metrics to define a conformal invariant called the Gauduchon degree and showed that if a metric ω has a non-positive Gauduchon degree then the Chern–Yamabe problem admits a solution.

From an algebraic perspective, singularities are ubiquitous as they occur in various situations, notably in the Minimal Model Program and moduli theory. Ueno [Uen75] found a birational class of 3-dimensional complex manifolds which does not admit a smooth minimal model. In moduli theory, it is necessary to deal with singular varieties when compactifying moduli spaces of smooth manifolds. Already in dimension one, the fundamental domain of moduli space of elliptic curves is non-compact and nodal curves lie on its boundary. Due to these considerations, singular version of canonical metric problems have been intensively studied on Kähler spaces or reflexive sheaves in recent decades. For example, generalizing Yau's theorem, singular Kähler–Einstein metrics on mildly singular Kähler varieties have been constructed in [EGZ09] and further studied by many authors (see [GZ17, Bou18, Don18] and the reference therein). On the other hand, Bando–Siu [BS94] generalized Uhlenbeck–Yau's theorem to reflexive sheaves over Kähler manifolds. Recently, Uhlenbeck–Yau's theorem was generalized on normal Kähler spaces.

Main result

Due to the above considerations, it is legitimate to look for singular Gauduchon metrics on singular hermitian varieties (irreducible, reduced, complex analytic spaces). This is a continuation of our previous work [Pan22] in which we raised the following question:

Question. Suppose that X is a compact complex variety equipped with a hermitian metric ω. Can one find a Gauduchon metric ω_G in the conformal class of ω?

In [Pan22, Theorem A], we showed the existence of bounded Gauduchon metrics on *smoothable* hermitian varieties. The purpose of this note is to provide an answer to the problem in full generally based on a conjecture about the existence of uniform Sobolev constant in [Ou22, Conj. 1.2]:

Conjecture (SC). *Let (V,ω) be an n-dimensional compact hermitian variety. Let $p : \widehat{V} \to V$ be a blowup with a smooth center. Fix $\widehat{\omega}$ a hermitian metric on $\widehat{V}$ and define*

$$\omega_\varepsilon := p^*\omega + \varepsilon\widehat{\omega}$$

for $\varepsilon \in (0,1]$. For all $1 \leq p < 2n$, there is a uniform constant $C_{S,p} > 0$ such that for all $\varepsilon \in (0,1]$ and for all $f \in L_1^p(\widehat{V}^{\mathrm{reg}}, \omega_\varepsilon)$,

$$\left(\int_{\widehat{V}} |f|^{\frac{2np}{2n-p}} \omega_\varepsilon^n\right)^{\frac{2n-p}{2np}} \leq C_{S,p} \left(\int_{\widehat{V}} |\mathrm{d}f|^p_{\omega_\varepsilon} \omega_\varepsilon^n + \int_{\widehat{V}} |f|^p \omega_\varepsilon^n\right)^{\frac{1}{p}}.$$

Theorem 3.0.1. *Assume that Conjecture (SC) holds. Let (X, ω) be a compact hermitian variety of dimension $n \geq 2$. There exists a bounded smooth function ρ on X^{reg} such that $\mathrm{dd^c}(\rho\omega^{n-1}) = 0$ on X^{reg} and ρ is bounded away from zero (i.e. $\rho \geq c > 0$ for some constant c). Such a function ρ is unique up to multiplication by a positive constant and $\rho^{\frac{1}{n-1}}\omega$ is a Gauduchon metric conformally equivalent to ω on X^{reg}.*

In [Ou22, Prop. 4.10], Ou proved that Conjecture (SC) holds when $p : \widehat{V} \to V$ is a blowup with one point.

Application in families

Let $(\mathcal{X}, \omega)$ be an $(n+1)$-dimensional hermitian variety and $\pi : \mathcal{X} \to \mathbb{D}$ be a proper surjective holomorphic map with connected fibres. Suppose that each fibre $X_t = \pi^{-1}(t)$ is a variety (irreducible, reduced). The metric ω induces a hermitian metric ω_t on every fibre X_t by restriction. According to Theorem 3.0.1, on each fibre X_t, there is a normalized Gauduchon factor ρ_t with respect to ω_t. We prove that such a function is uniformly bounded in families:

Proposition 3.0.2. *Let $\pi : (\mathcal{X}, \omega) \to \mathbb{D}$ be a family of hermitian varieties. Assume that (X_t, ω_t) admits a bounded Gauduchon factor for all $t \in \mathbb{D}$. Then there is a uniform upper bound C_G of the normalized Gauduchon factors ρ_t on each fibre X_t for all $t \in \overline{\mathbb{D}}_{1/2}$.*

Using the uniform estimate of normalized bounded Gauduchon factors $(\rho_t)_t$ in Proposition 3.0.2, we generalize the Sup-L^1 comparison [DGG20, Proposition 3.3] in families of hermitian varieties where all fibres have only isolated singularities:

Proposition 3.0.3. *Let $\pi : (\mathcal{X}, \omega) \to \mathbb{D}$ be a family of hermitian varieties. Suppose that each fibre X_t has only isolated singularities and (X_t, ω_t) admits a bounded Gauduchon factor. Then there is a constant $C_{SL} > 0$ such that for all $t \in \overline{\mathbb{D}}_{1/2}$,*

$$\forall \varphi_t \in \mathrm{PSH}(X_t, \omega_t) \quad \sup_{X_t} \varphi_t - C_{SL} \leq \frac{1}{V_t} \int_{X_t} \varphi_t \omega_t^n \leq \sup_{X_t} \varphi_t$$

where V_t is the volume of X_t with respect to ω_t.

This generalization removes the *smoothing* assumption in [Pan23, Proposition B], and supports the Sup-L^1 comparison conjecture [DGG20, Conjecture 3.1] in families of hermitian varieties.

Organization

In Section 3.1, we establish a Harnack-type inequality (Theorem 3.1.1) with respect to the elliptic operator Δ_ω^* on singular hermitian varieties. In Section 3.2, we keep track of the constants arising in Theorem 3.1.1 under resolution of singularities and show Theorem 3.0.1. In Section 3.3, we consider families of hermitian varieties and prove Proposition 3.0.2 and Proposition 3.0.3.

3.1 An a priori estimate

We first recall the notation $\mathrm{d^c} = \frac{\mathrm{i}}{2}(\bar\partial - \partial)$ so that $\mathrm{dd^c} = \mathrm{i}\partial\bar\partial$. We denote by

$$\Delta_\omega f := \frac{n\mathrm{dd^c} f \wedge \omega^{n-1}}{\omega^n}, \quad |\mathrm{d}f|^2_\omega := \frac{n\mathrm{d}f \wedge \mathrm{d^c} f \wedge \omega^{n-1}}{\omega^n}, \quad \text{and} \quad \Delta^*_\omega f := \frac{n\mathrm{dd^c}(f\omega^{n-1})}{\omega^n}.$$

.

Fix a compact hermitian variety (X, ω) of complex dimension n. Let $B \geq 0$ be a constant such that $-B\omega^n \leq \mathrm{dd^c}\omega^{n-1} \leq B\omega^n$ and $V := \mathrm{Vol}_\omega(X)$. We shall explain a Harnack inequality which generalizes [Pan22, Theorem 2.1]:

Theorem 3.1.1. *If ρ is a normalized bounded Gauduchon factor (i.e. $\mathrm{dd^c}\left(\rho\omega^{n-1}\right) = 0$ on X^{reg}, $\rho \in L^\infty(X^{\mathrm{reg}})$, and $\inf_{X^{\mathrm{reg}}} \rho = 1$), then there is a constant C_G depending only on n, B, C_S, C_P and V such that*

$$\sup_{X^{\mathrm{reg}}} \rho \leq C_G,$$

where C_S and C_P are Sobolev and Poincaré constants such that

$$\forall f \in L^2_1(X^{\mathrm{reg}}), \quad \left(\int_X |f|^{\frac{2n}{n-1}}\omega^n\right)^{\frac{n-1}{n}} \leq C_S\left(\int_X |\mathrm{d}f|^2_\omega\omega^n + \int_X |f|^2\omega^n\right)$$

and

$$\forall f \in L^2_1(X^{\mathrm{reg}}) \text{ and } \int_X f\omega^n = 0, \quad \int_X |f|^2\omega^n \leq C_P \int_X |\mathrm{d}f|^2_\omega \omega^n.$$

3.1.1 Proof of Theorem 3.1.1

Good cut-off functions

First of all, we recall the existence of cut-off functions with small L^2-gradients. From a classical property in Riemannian geometry, these cut-off functions do exist on so-called *parabolic* manifolds (see eg. [Gla83, EG92], or page 35 for more information). In our case, one can easily construct explicit cut-off functions by using Hironaka's desingularization and log log-potentials (cf. [Ber12, Lemma 2.2] and [CGP13, Section 9]):

Lemma 3.1.2. *Suppose that (X, ω) is a compact hermitian variety. There exist cut-off functions $(\chi_\delta)_{\delta>0} \subset \mathcal{C}^\infty_c(X^{\mathrm{reg}})$ satisfying the following properties*

(i) χ_δ *is increasing to the characteristic function of* X^{reg} *when* δ *decreases to* 0*;*

(ii) $\int_X |\mathrm{dd^c}\chi_\delta \wedge \omega^{n-1}| \to 0$ *when* $\delta \to 0$*;*

(iii) $\int_X \mathrm{d}\chi_\delta \wedge \mathrm{d^c}\chi_\delta \wedge \omega^{n-1} \to 0$ *when* $\delta \to 0$

These functions allow us to perform integration by parts as on a compact manifold. Using these cut-off functions, we establish a lemma similar to [Pan22, Lemma 2.4] on the regular locus of X:

Lemma 3.1.3. *Suppose that ρ is a positive smooth function on X^{reg}, ρ is bounded away from zero and infinity, and $\rho \in \ker \Delta_\omega^*$, i.e. $\mathrm{dd^c}(\rho\omega^{n-1}) = 0$. Then we have*

$$\int_{X^{\mathrm{reg}}} F'(\rho)\mathrm{d}\rho \wedge \mathrm{d^c}\rho \wedge \omega^{n-1} = \int_{X^{\mathrm{reg}}} G(\rho)\mathrm{dd^c}\omega^{n-1}$$

where F and G are two real $\mathcal{C}^1$-functions defined on $\mathbb{R}_{>0}$ which satisfy $xF'(x) = G'(x)$.

Remark 3.1.4. One the LHS, the integration $\int_{X^{\mathrm{reg}}} F'(\rho)\mathrm{d}\rho \wedge \mathrm{d^c}\rho \wedge \omega^{n-1}$ is well-defined for $F \in \mathcal{C}^1(\mathbb{R}_{>0}, \mathbb{R})$. The reason is as follows: We should first assume that $F'(\rho)$ has a sign. Taking $F'(x) = 1$ and $G(x) = \frac{x^2}{2}$, we have $\int_{X^{\mathrm{reg}}} \mathrm{d}\rho \wedge \mathrm{d^c}\rho \wedge \omega^{n-1} = \frac{1}{2}\int_{X^{\mathrm{reg}}} \rho^2\mathrm{dd^c}\omega^{n-1}$ which is bounded, so one finds $\mathrm{d}\rho \in L^2$ and thus it is legitimate to define $\int_{X^{\mathrm{reg}}} F'(\rho)\mathrm{d}\rho \wedge \mathrm{d^c}\rho \wedge \omega^{n-1}$ for F is merely a $\mathcal{C}^1$-function.

Proof. Since $\rho \in \ker \Delta_\omega^*$, we have $\mathrm{dd^c}(\rho\omega^{n-1}) = 0$. Let χ_δ be a cut-off function introduced in Lemma 3.1.2. Using Stokes' theorem, it follows that

$$\begin{aligned}
0 &= \int_{X^{\mathrm{reg}}} \chi_\delta F(\rho)\mathrm{dd^c}(\rho\omega^{n-1}) = -\int_{X^{\mathrm{reg}}} \mathrm{d}\left(\chi_\delta F(\rho)\right)\mathrm{d^c}(\rho\omega^{n-1}) \\
&= -\int_{X^{\mathrm{reg}}} F(\rho)\mathrm{d}\chi_\delta \wedge \mathrm{d^c}(\rho\omega^{n-1}) - \int_{X^{\mathrm{reg}}} \chi_\delta F'(\rho)\mathrm{d}\rho \wedge \mathrm{d^c}(\rho\omega^{n-1}) \\
&= -\int_{X^{\mathrm{reg}}} F(\rho)\mathrm{d}\chi_\delta \wedge \mathrm{d^c}\rho \wedge \omega^{n-1} - \int_{X^{\mathrm{reg}}} \rho F(\rho)\mathrm{d}\chi_\delta \wedge \mathrm{d^c}\omega^{n-1} \\
&\quad - \int_{X^{\mathrm{reg}}} \chi_\delta F'(\rho)\mathrm{d}\rho \wedge \mathrm{d^c}\rho \wedge \omega^{n-1} - \int_{X^{\mathrm{reg}}} \chi_\delta \rho F'(\rho)\mathrm{d}\rho \wedge \mathrm{d^c}\omega^{n-1}.
\end{aligned}$$

By the assumption $xF'(x) = G'(x)$ and direct computations, we derive

$$\int_{X^{\mathrm{reg}}} \chi_\delta F'(\rho)\mathrm{d}\rho \wedge \mathrm{d^c}\rho \wedge \omega^{n-1} = \int_{X^{\mathrm{reg}}} \chi_\delta G(\rho)\mathrm{dd^c}\omega^{n-1} + \int_{X^{\mathrm{reg}}} \left(\rho F(\rho) - G(\rho)\right)\mathrm{dd^c}\chi_\delta \wedge \omega^{n-1}. \tag{3.1.1}$$

Indeed,

$$\begin{aligned}
&\int_{X^{\mathrm{reg}}} \chi_\delta F'(\rho)\mathrm{d}\rho \wedge \mathrm{d}^c\rho \wedge \omega^{n-1} \\
&= -\int_{X^{\mathrm{reg}}} F(\rho)\mathrm{d}\chi_\delta \wedge \mathrm{d}^c\rho \wedge \omega^{n-1} - \int_{X^{\mathrm{reg}}} \rho F(\rho)\mathrm{d}\chi_\delta \wedge \mathrm{d}^c\omega^{n-1} - \int_{X^{\mathrm{reg}}} \chi_\delta G'(\rho)\mathrm{d}\rho \wedge \mathrm{d}^c\omega^{n-1} \\
&= -\int_{X^{\mathrm{reg}}} F(\rho)\mathrm{d}\rho \wedge \mathrm{d}^c\chi_\delta \wedge \omega^{n-1} - \int_{X^{\mathrm{reg}}} \rho F(\rho)\mathrm{d}\chi_\delta \wedge \mathrm{d}^c\omega^{n-1} - \int_{X^{\mathrm{reg}}} \chi_\delta \mathrm{d}G(\rho) \wedge \mathrm{d}^c\omega^{n-1} \\
&= \int_{X^{\mathrm{reg}}} \rho F'(\rho)\mathrm{d}\rho \wedge \mathrm{d}^c\chi_\delta \wedge \omega^{n-1} + \int_{X^{\mathrm{reg}}} \rho F(\rho)\mathrm{d}\mathrm{d}^c\chi_\delta \wedge \omega^{n-1} - \int_{X^{\mathrm{reg}}} \rho F(\rho)\mathrm{d}^c\chi_\delta \wedge \mathrm{d}\omega^{n-1} \\
&\quad - \int_{X^{\mathrm{reg}}} \rho F(\rho)\mathrm{d}\chi_\delta \wedge \mathrm{d}^c\omega^{n-1} \\
&\quad + \int_{X^{\mathrm{reg}}} \chi_\delta G(\rho)\mathrm{d}\mathrm{d}^c\omega^{n-1} + \int_{X^{\mathrm{reg}}} G(\rho)\mathrm{d}\chi_\delta \wedge \mathrm{d}^c\omega^{n-1} \\
&= \int_{X^{\mathrm{reg}}} \mathrm{d}G(\rho) \wedge \mathrm{d}^c\chi_\delta \wedge \omega^{n-1} + \int_{X^{\mathrm{reg}}} \rho F(\rho)\mathrm{d}\mathrm{d}^c\chi_\delta \wedge \omega^{n-1} \\
&\quad + \int_{X^{\mathrm{reg}}} \chi_\delta G(\rho)\mathrm{d}\mathrm{d}^c\omega^{n-1} + \int_{X^{\mathrm{reg}}} G(\rho)\mathrm{d}\chi_\delta \wedge \mathrm{d}^c\omega^{n-1} \\
&= -\int_{X^{\mathrm{reg}}} G(\rho)\mathrm{d}\mathrm{d}^c\chi_\delta \wedge \omega^{n-1} + \int_{X^{\mathrm{reg}}} G(\rho)\mathrm{d}^c\chi_\delta \wedge \mathrm{d}\omega^{n-1} + \int_{X^{\mathrm{reg}}} \rho F(\rho)\mathrm{d}\mathrm{d}^c\chi_\delta \wedge \omega^{n-1} \\
&\quad + \int_{X^{\mathrm{reg}}} \chi_\delta G(\rho)\mathrm{d}\mathrm{d}^c\omega^{n-1} + \int_{X^{\mathrm{reg}}} G(\rho)\mathrm{d}\chi_\delta \wedge \mathrm{d}^c\omega^{n-1} \\
&= -\int_{X^{\mathrm{reg}}} G(\rho)\mathrm{d}\mathrm{d}^c\chi_\delta \wedge \omega^{n-1} + \int_{X^{\mathrm{reg}}} \rho F(\rho)\mathrm{d}\mathrm{d}^c\chi_\delta \wedge \omega^{n-1} + \int_{X^{\mathrm{reg}}} \chi_\delta G(\rho)\mathrm{d}\mathrm{d}^c\omega^{n-1}.
\end{aligned}$$

Since ρ is bounded from above and away from zero, and F, G are continuous, the function $\rho F(\rho) - G(\rho)$ is bounded on X^{reg}. By Lemma 3.1.2, taking the limit of $\delta \to 0$ on both sides of (3.1.1), one can conclude the proof. □

Harnack inequality

One can now proceed as in the proof of [Pan22, Theorem 2.1] to establish Theorem 3.1.1.

Sketch of proof of Theorem 3.1.1. As Lemma 3.1.3 generalizes [Pan22, Lemma 2.4], one can choose same test functions and follow the same computations as in [Pan22, page 7-10]. □

3.2 Singular Gauduchon metrics

Suppose that (X, ω) is an n-dimensional compact hermitian variety. Using Hironaka's theorem [Hir64], one can find a resolution of singularities $\mu : \widetilde{X} \to X$ which is a composition of finitely many blowups with smooth centers. Namely, we can express μ by

$$\widetilde{X} = X_m \xrightarrow{p_m} X_{m-1} \xrightarrow{p_{m-1}} \cdots \xrightarrow{p_2} X_1 \xrightarrow{p_1} X_0 = X$$

where $p_k : X_k \to X_{k-1}$ is a blowup with smooth center. Since the exceptional divisor $\mathrm{Exc}(p_k)$ is relatively anti-ample for all k, for all hermitian metric ω_{k-1} on X_{k-1}, there is a closed $(1,1)$-form θ_k such that $p_k^*\omega_{k-1} + \theta_k$ is a hermitian metric on X_k. Inductively, we find $(1,1)$-forms $(\theta_j)_{j\in\{1,\cdots,m\}}$ such that for all $k \in \{1,\cdots,m\}$

$$\omega_k := \omega + \sum_{j=1}^{k} \theta_j$$

defines a hermitian metric on X_k. By abuse of notation, here and hereafter we denote the geometric objects and their pullback by the same notation very often. One should be aware that ω is merely semi-positive on X_k for all $k > 0$. Let $\varepsilon(k) := (\varepsilon_1, \cdots, \varepsilon_k) \in (0,1]^k$. For $k \in \{1,\cdots,m\}$, we define inductively a family of hermitian metrics on X_k by

$$\omega_{\varepsilon(k)} := \omega_{\varepsilon(k-1)} + \varepsilon_k \underbrace{\left(\omega_{\varepsilon(k-1)} + \left[\prod_{j=0}^{k-1} \varepsilon_j\right] \theta_k\right)}_{=:\widehat{\omega}_{\varepsilon(k-1)}}$$

where $\omega_{\varepsilon(0)} := \omega$ and $\varepsilon_0 := 1$. By construction, one can derive that

- $\omega_{\varepsilon(k)} \geq \varepsilon_k \widehat{\omega}_{\varepsilon(k-1)} = \varepsilon_k \left(\omega_{\varepsilon(k-1)} + \left[\prod_{j=0}^{k-1} \varepsilon_j\right] \theta_k\right) \geq \left(\prod_{j=0}^{k} \varepsilon_j\right) \omega_k$ which is positive on X_k so is a hermitian metric on X_k;
- $\omega_{\varepsilon(k)}$ converges to $\omega_{\varepsilon(k-1)}$ when ε_k tends to 0.

We shall prove Theorem 3.0.1 by induction on k. We now focus on the following setup:

Setup (G). *Let (V, ω_V) be a compact hermitian variety. Consider*

- *$p : \widehat{V} \to V$ a blowup of a smooth center in V^{sing};*
- *$\widehat{\omega}$ a hermitian metric on $\widehat{V}$ such that $\widehat{\omega} = p^*\omega_V + \theta$ where θ is a closed $(1,1)$-form.*

For $\varepsilon \in (0,1]$, we define

$$\omega_\varepsilon := p^*\omega_V + \varepsilon\widehat{\omega}$$

which forms a family of hermitian metrics on $\widehat{V}$.

Proposition 3.2.1. *Under Setting (G), suppose that for all $\varepsilon \in (0,1]$, there exists a normalized bounded Gauduchon factor ρ_ε on $\widehat{V}^{\mathrm{reg}}$ with respect to the hermitian metric ω_ε. Then there is a uniform constant C_G such that for all $\varepsilon \in (0,1]$*

$$\sup_{\widehat{V}^{\mathrm{reg}}} \rho_\varepsilon \leq C_G.$$

3.2.1 Proof of Proposition 3.2.1

Once we find a uniform control on the constants $B(\omega_\varepsilon)$, $\mathrm{Vol}_{\omega_\varepsilon}(\widehat{V})$, $C_S(\omega_\varepsilon)$, and $C_P(\omega_\varepsilon)$ for all $\varepsilon \in (0,1]$, Proposition 3.2.1 would be a direct consequence of Theorem 3.1.1.

Controlling $B(\omega_\varepsilon)$ and the volumes $\mathrm{Vol}_{\omega_\varepsilon}(\widehat{V})$

On (V, ω_V), there is a constant $B' > 0$ such that

$$-B'\omega_V^2 \leq dd^c\omega_V \leq B'\omega_V^2 \quad \text{and} \quad -B'\omega_V^3 \leq d\omega_V \wedge d^c\omega_V \leq B'\omega_V^3. \tag{3.2.1}$$

Using binomial expansions, we obtain

$$dd^c\omega_\varepsilon^{n-1} = \sum_{j=0}^{n-1} \varepsilon^j \binom{n-1}{j} \underbrace{dd^c\left(\omega_V^{n-1-j} \wedge \widehat{\omega}^j\right)}_{=:\alpha_j},$$

and

$$\omega_\varepsilon^n = \sum_{j=0}^{n} \varepsilon^j \binom{n}{j} \underbrace{\omega_V^{n-j} \wedge \widehat{\omega}^j}_{=:\beta_j \geq 0}.$$

Hence, to search a constant $B > 0$ satisfying $-B\omega_\varepsilon^n \leq dd^c\omega_\varepsilon^{n-1} \leq B\omega_\varepsilon^n$, it is enough to prove that for all $j \in \{0, \cdots, n-1\}$, one has $-\beta_j \lesssim \alpha_j \lesssim \beta_j$. Note that

$$\begin{aligned}
\alpha_j &= dd^c\left(\omega_V^{n-1-j} \wedge \widehat{\omega}^j\right) \\
&= \underbrace{dd^c\omega_V^{n-1-j} \wedge \widehat{\omega}^j}_{=:\alpha_{j,1}} + \underbrace{\omega_V^{n-1-j} \wedge dd^c\widehat{\omega}^j}_{=:\alpha_{j,2}} + 2\underbrace{d\omega_V^{n-1-j} \wedge d^c\widehat{\omega}^j}_{=:\alpha_{j,3}}.
\end{aligned}$$

Since $d\widehat{\omega} = d\omega_V$, $d^c\widehat{\omega} = d^c\omega_V$ and $dd^c\widehat{\omega} = dd^c\omega_V$, we have

$$\begin{aligned}
\alpha_{j,1} &= (n-1-j)\left(dd^c\omega_V \wedge \omega_V^{n-2-j}\right) \wedge \widehat{\omega}^j \\
&\quad + (n-1-j)(n-2-j)\left(d\omega_V \wedge d^c\omega_V \wedge \omega_V^{n-3-j}\right) \wedge \widehat{\omega}^j \\
\alpha_{j,2} &= \underbrace{j\left(dd^c\omega_V \wedge \omega_V^{n-1-j}\right) \wedge \widehat{\omega}^{j-1}}_{=:\alpha'_{j,2}} \\
&\quad + \underbrace{j(j-1)\left(d\omega_V \wedge d^c\omega_V \wedge \omega_V^{n-1-j}\right) \wedge \widehat{\omega}^{j-2}}_{=:\alpha''_{j,2}} \\
\alpha_{j,3} &= j(n-1-j)\left(d\omega_V \wedge d^c\omega_V \wedge \omega_V^{n-2-j}\right) \wedge \widehat{\omega}^{j-1}.
\end{aligned}$$

Using (3.2.1), one can deduce that there is a constant $D = D(n, B')$ such that for all j,

$$-D\beta_j \leq \alpha_{j,1} \leq D\beta_j, \quad -D\beta_{j-2} \leq \alpha'_{j,2}, \alpha_{j,3} \leq D\beta_{j-2}, \quad -D\beta_{j-1} \leq \alpha''_{j,2} \leq D\beta_{j-1}$$

Note that the power of ε in front of $\alpha_{j,1}$ (resp. $\alpha'_{j,2}$, resp. $\alpha''_{j,2}, \alpha_{j,3}$) is less than or equal to the power of ε in front of β_j (resp. β_{j-2}, resp. β_{j-1}). Thus, there exists a uniform constant $B > 0$

which depends only on n, B' such that for all $\varepsilon \in (0,1]$,

$$-B\omega_\varepsilon^n \leq \mathrm{dd^c}\omega_\varepsilon^{n-1} \leq B\omega_\varepsilon^n. \tag{3.2.2}$$

Note that $\omega \leq \omega_\varepsilon \leq \omega + \widehat{\omega}$ and ω is positive on $\widehat{V} \setminus \mathrm{Exc}(p)$. Thus, one can find a uniform constant $C_{\mathrm{Vol}} > 0$ satisfying

$$\forall \varepsilon \in (0,1], \quad C_{\mathrm{Vol}}^{-1} \leq \mathrm{Vol}_{\omega_\varepsilon}(\widehat{V}) \leq C_{\mathrm{Vol}}. \tag{3.2.3}$$

Uniform Sobolev and Poincaré constants

In this section, based on Conjecture (SC), we shall show the existence of uniform Poincaré constants for $(\widehat{V}, \omega_\varepsilon)_{\varepsilon\in(0,1]}$.

Then arguing as in [RZ11, Proposition 3.2] and using the irreducibility of V, one can derive uniform Poincaré inequalities:

Lemma 3.2.2. *Assume that Conjecture (SC) is true. Under Setting (G), there is a uniform constant $C_P > 0$ such that for all $\varepsilon \in (0,1]$ and for all $f \in L_1^2(\widehat{V}^{\mathrm{reg}}, \omega_\varepsilon)$ with $\int_{\widehat{V}} f\omega_\varepsilon^n = 0$,*

$$\int_{\widehat{V}} |f|^2 \omega_\varepsilon^n \leq C_P \int_{\widehat{V}} |\mathrm{d}f|_{\omega_\varepsilon}^2 \,\omega_\varepsilon^n.$$

Proof. First, we claim that for each $\varepsilon \in (0,1]$, there exists such a Poincaré constant $C_{P,\varepsilon}$. Indeed, thanks to [Bei19, Theorem 0.1 (3)], for each $\varepsilon \in (0,1]$, the inclusion $L_1^2(\widehat{V}^{\mathrm{reg}}, \omega_\varepsilon) \hookrightarrow L^2(\widehat{V}^{\mathrm{reg}}, \omega_\varepsilon)$ is compact. Then following the standard argument of the Poincaré–Wirtinger inequality (cf. [Eva10, page 290]) and the irreducibility of $\widehat{V}$, one gets such a constant $C_{P,\varepsilon}$.

Now, we closely follow the idea of [RZ11, Proposition 3.2] to show the existence of a uniform Poincaré constant. Suppose that a uniform constant C_P does not exist. Namely, there exists a sequence of $L_1^2(\widehat{V}, \omega_{\varepsilon_j})$-functions $(\varphi_j)_j$ satisfying

$$\int_{\widehat{V}} \varphi_j \omega_{\varepsilon_j}^n = 0, \quad \int_{\widehat{V}} |\varphi_j|^2 \omega_{\varepsilon_j}^n = 1, \quad \text{and} \int_{\widehat{V}} |\mathrm{d}\varphi_j|_{\omega_{\varepsilon_j}}^2 \omega_{\varepsilon_j}^n \leq 1/j.$$

We want to draw a contradiction. Without loss of generality, we may assume that $\varepsilon_j \to 0$ as $j \to +\infty$. Note that for any compact subset $K \subset \widehat{V}^{\mathrm{reg}} \setminus \mathrm{Exc}(p)$, the metric ω_ε converges smoothly to the metric $p^*\omega_V$ on a neighborhood of K in $\widehat{V}^{\mathrm{reg}} \setminus \mathrm{Exc}(p)$; hence for all $\varepsilon \in (0,1]$, ω_ε is equivalent to $p^*\omega_V$ on K. Now, we fix a connected compact subset $K \subset \widehat{V}^{\mathrm{reg}} \setminus \mathrm{Exc}(p)$. The assumptions imply that $(\varphi_j)_j$ is a bounded sequence in $L_1^2(K)$. By Banach–Alaoglu theorem, there exists a $L_1^2(K)$-function φ such that φ_j converges to φ weakly in $L_1^2(K)$. Moreover, φ_j converges to φ strongly in $L^2(K)$ by Rellich theorem.

By lower semi-continuity, we have

$$0 \leq \|\mathrm{d}\varphi\|_{L^2(K)} \leq \varliminf_{j\to\infty} \|\mathrm{d}\varphi_j\|_{L^2(K)} = 0.$$

Hence, φ is a locally constant function on K and thus it is a constant by connectedness. Using Hölder inequality, one can derive

$$\left|\int_{\widehat{V}\setminus K}\varphi_j\omega_{\varepsilon_j}^n\right| \leq \mathrm{Vol}_{\omega_{\varepsilon_j}}^{\frac{1}{2}}(\widehat{V}\setminus K)\,\|\varphi_j\|_{L^2(\widehat{V}\setminus K)} \leq \mathrm{Vol}_{\omega_{\varepsilon_j}}^{\frac{1}{2}}(\widehat{V}\setminus K). \tag{3.2.4}$$

Note that

$$\left|\int_K \varphi_j\omega_{\varepsilon_j}^n\right| = \left|\int_{\widehat{V}\setminus K}\varphi_j\omega_{\varepsilon_j}^n\right| \tag{3.2.5}$$

since $\int_{\widehat{V}}\varphi_j\omega_{\varepsilon_j}^n = 0$. Using that φ is constant and applying (3.2.4) and (3.2.5), we obtain

$$\begin{aligned}\mathrm{Vol}_{p^*\omega_V}(K)|\varphi| = \left|\int_K \varphi(p^*\omega_V)^n\right| &= \lim_{j\to\infty}\left|\int_K \varphi_j\omega_{\varepsilon_j}^n\right| = \lim_{j\to\infty}\left|\int_{\widehat{V}\setminus K}\varphi_j\omega_{\varepsilon_j}^n\right| \\ &\leq \lim_{j\to\infty}\mathrm{Vol}_{\omega_{\varepsilon_j}}^{\frac{1}{2}}(\widehat{V}\setminus K) = \mathrm{Vol}_{p^*\omega_V}^{\frac{1}{2}}(\widehat{V}\setminus K).\end{aligned} \tag{3.2.6}$$

On the other hand, using Conjecture (SC), it follows that

$$\left(\int_{\widehat{V}}|\varphi_j|^{\frac{2n}{n-1}}\omega_{\varepsilon_j}^n\right)^{\frac{n-1}{n}} \leq C_S\left(\int_{\widehat{V}}|\mathrm{d}\varphi_j|_{\omega_{\varepsilon_j}}^2\omega_{\varepsilon_j}^n + \int_{\widehat{V}}|\varphi_j|^2\omega_{\varepsilon_j}^n\right) \leq C_S\,(1/j+1) \leq 2C_S.$$

By Hölder inequality, one can derive

$$\int_{\widehat{V}\setminus K}|\varphi_j|^2\omega_{\varepsilon_j}^n \leq \left(\int_{\widehat{V}}|\varphi_j|^{\frac{2n}{n-1}}\omega_{\varepsilon_j}^n\right)^{\frac{n-1}{n}}\mathrm{Vol}_{\omega_{\varepsilon_j}}^{\frac{1}{n}}(\widehat{V}\setminus K) \leq 2C_S\,\mathrm{Vol}_{\omega_{\varepsilon_j}}^{\frac{1}{n}}(\widehat{V}\setminus K). \tag{3.2.7}$$

Thus,

$$\begin{aligned}1 &= \lim_{j\to\infty}\int_{\widehat{V}}|\varphi_j|^2\omega_{\varepsilon_j}^n = \lim_{j\to\infty}\int_{\widehat{V}\setminus K}|\varphi_j|^2\omega_{\varepsilon_j}^n + \lim_{j\to\infty}\int_K|\varphi_j|^2\omega_{\varepsilon_j}^n \\ &\leq 2C_S\,\mathrm{Vol}_{p^*\omega_V}^{\frac{1}{n}}(\widehat{V}\setminus K) + \int_K|\varphi|^2\omega^n = 2C_S\,\mathrm{Vol}_{p^*\omega_V}^{\frac{1}{n}}(\widehat{V}\setminus K) + \mathrm{Vol}_{p^*\omega_V}(K)|\varphi|^2 \\ &\leq 2C_S\,\mathrm{Vol}_{p^*\omega_V}^{\frac{1}{n}}(\widehat{V}\setminus K) + \frac{\mathrm{Vol}_{p^*\omega_V}(\widehat{V}\setminus K)}{\mathrm{Vol}_{p^*\omega_V}(K)}.\end{aligned} \tag{3.2.8}$$

Here the first inequality comes from (3.2.7) and second inequality is referred to (3.2.6). Since $\widehat{V}^{\mathrm{reg}}\setminus\mathrm{Exc}(p)$ is connected by the irreducibility of V, we can choose a sequence of connected compact subset $K_j\subset\widehat{V}^{\mathrm{reg}}\setminus\mathrm{Exc}(p)$ such that $K_j\subset K_{j+1}$ for all j and $\mathrm{Vol}_{p^*\omega_V}(\widehat{V}\setminus K_j)\xrightarrow[j\to+\infty]{}0$. Then the RHS of (3.2.8) tends to zero and this yields a contradiction. □

Proposition 3.2.1 follows now from Theorem 3.1.1, (3.2.2), (3.2.3), Conjecture (SC) and 3.2.2.

3.2.2 Proof of Theorem 3.0.1

We are now going to construct a bounded Gauduchon factor on X^{reg} by induction. The first step consists in constructing bounded Gauduchon factors when we blow down a variety. We closely follow the strategy in our previous work [Pan22, Theorem 3.1].

Proposition 3.2.3. *Under Setting (G) and suppose that for all $\varepsilon \in (0,1]$, there exists a normalized bounded Gauduchon factor ρ_ε on $\widehat{V}^{\mathrm{reg}}$ with respect to the hermitian metric ω_ε. Then there is a function $\rho \in \mathcal{C}^\infty(V^{\mathrm{reg}})$ such that $1 \leq \rho \leq C_G$ and $\mathrm{dd^c}(\rho\omega_V^{n-1}) = 0$ on V^{reg}.*

Proof. We shall apply standard elliptic theory on some relatively compact subsets of $\widehat{V}^{\mathrm{reg}} \setminus \mathrm{Exc}(p)$ to get a smooth function ρ. This ρ is the limit of $(\rho_{\varepsilon_j})_{j\in\mathbb{N}}$ for a sequence $\varepsilon_j \to 0$ when $j \to +\infty$.

We set $P_\varepsilon = \Delta^*_{\omega_\varepsilon}$ and fix $U_1 \Subset U_2 \Subset \widehat{V}^{\mathrm{reg}} \setminus \mathrm{Exc}(p)$ which are connected open subsets. On U_2, the Riemannian metric g_0 induced by $p^*\omega_V$ is quasi-isometric to the metric g_ε induced by ω_ε, and the volume form $p^*\omega_V^n$ is comparable with ω_ε^n. In other words, we have a uniform constant $C_{U_2} > 0$ such that for all $\varepsilon \in (0,1]$,

$$C_{U_2}^{-1}\langle\cdot,\cdot\rangle_{p^*\omega_V} \leq \langle\cdot,\cdot\rangle_{\omega_\varepsilon^n} \leq C_{U_2}\langle\cdot,\cdot\rangle_{p^*\omega_V}, \text{ and } C_{U_2}^{-1}p^*\omega_V^n \leq \omega_\varepsilon^n \leq C_{U_2}p^*\omega_V^n \tag{3.2.9}$$

on U_2.

By Gårding inequality, we have

$$\|u\|_{L^2_2(U_1,p^*\omega_V)} \leq C_{U_1,U_2}\left(\|P_\varepsilon u\|_{L^2(U_2,p^*\omega_V)} + \|u\|_{L^2(U_2,p^*\omega_V)}\right)$$

for all $u \in \mathcal{C}^\infty_c(U_2)$. The constant C_{U_1,U_2} can be chosen independent of ε because the coefficients of P_ε move smoothly in ε on $\overline{U_2}$. Choose a cut-off function χ such that $\mathrm{supp}(\chi) \subset U_2$ and $\chi \equiv 1$ on U_1. From Gårding inequality, it follows that

$$\|\rho_\varepsilon\|_{L^2_2(U_1,p^*\omega_V)} \leq C_{U_1,U_2}\left(\|P_\varepsilon(\chi\rho_\varepsilon)\|_{L^2(U_2,p^*\omega_V)} + \|\chi\rho_\varepsilon\|_{L^2(U_2,p^*\omega_V)}\right). \tag{3.2.10}$$

In (3.2.10), the second term $\|\chi\rho_\varepsilon\|_{L^2(U_2,p^*\omega_V)}$ is uniformly bounded thanks to Proposition 3.2.1. Hence, we only need to estimate $\|P_\varepsilon(\chi\rho_\varepsilon)\|_{L^2(U_2,p^*\omega_V)}$. Note that

$$\begin{aligned} P_\varepsilon(\chi\rho_\varepsilon) &= \frac{n}{\omega_\varepsilon^n}\mathrm{dd^c}(\chi\rho_\varepsilon\omega_\varepsilon^{n-1}) \\ &= \frac{n}{\omega_\varepsilon^n}\left(2\mathrm{d}\rho_\varepsilon \wedge \mathrm{d^c}\chi \wedge \omega_\varepsilon^{n-1} + \chi\mathrm{dd^c}(\rho_\varepsilon\omega_\varepsilon^{n-1}) + \rho_\varepsilon\mathrm{dd^c}(\chi\omega_\varepsilon^{n-1}) - \chi\rho_\varepsilon\mathrm{dd^c}\omega_\varepsilon^{n-1}\right) \\ &= 2\langle\mathrm{d}\rho_\varepsilon, \mathrm{d}\chi\rangle_{\omega_\varepsilon} + \rho_\varepsilon P_\varepsilon(\chi) - \rho_\varepsilon\chi\frac{\mathrm{dd^c}\omega_\varepsilon^n}{\omega_\varepsilon^n}. \end{aligned} \tag{3.2.11}$$

Obviously, $\rho_\varepsilon P_\varepsilon(\chi)$ and $\rho_\varepsilon\chi\frac{\mathrm{dd^c}\omega_\varepsilon^{n-1}}{\omega_\varepsilon^n}$ are uniformly bounded, so we only need to control the L^2-

norm of the first term $\langle \mathrm{d}\rho_\varepsilon, \mathrm{d}\chi\rangle_{\omega_\varepsilon}$:

$$\begin{aligned}\int_{U_2} \left|\langle \mathrm{d}\rho_\varepsilon, \mathrm{d}\chi\rangle_{\omega_\varepsilon}\right|^2 p^*\omega_V^n &\le C_{U_2}^2 \left(\sup_{U_2} |\mathrm{d}\chi|^2_{p^*\omega_V}\right) \int_{U_2} |\mathrm{d}\rho_\varepsilon|^2_{\omega_\varepsilon} \omega_\varepsilon^n \\ &\le C_{U_2}^2 \left(\sup_{U_2} |\mathrm{d}\chi|^2_{p^*\omega_V}\right) \int_{\widehat{V}} |\mathrm{d}\rho_\varepsilon|^2_{\omega_\varepsilon} \omega_\varepsilon^n \\ &\le C_{U_2}^2 \left(\sup_{U_2} |\mathrm{d}\chi|^2_{p^*\omega_V}\right) \frac{nB}{2} \int_{\widehat{V}} \rho_\varepsilon^2 \omega_\varepsilon^n.\end{aligned}$$

Here the first line is by Cauchy–Schwarz inequality and (3.2.9). The fourth line follows from Lemma 3.1.3 by taking $F'(x) = n$ and $G(x) = \frac{nx^2}{2}$. By Proposition 3.2.1 and (3.2.3), we have $1 \le \rho_\varepsilon \le C_G$ and $\mathrm{Vol}_{\omega_\varepsilon}(\widehat{V}) \le C_{\mathrm{Vol}}$. One can find a uniform bound of $\|P_\varepsilon(\chi\rho_\varepsilon)\|_{L^2(U_2, p^*\omega_V)}$. Hence, $\|\rho_\varepsilon\|_{L^2_2(U_1, p^*\omega_V)}$ is uniformly bounded by some uniform constant $C(U_1, U_2)$.

For higher order estimates, we apply higher order Gårding inequalities on the fixed domains $U_1 \Subset U_2 \Subset \widehat{V}^{\mathrm{reg}} \setminus \mathrm{Exc}(p)$:

$$\|u\|_{L^2_{s+2}(U_1, p^*\omega_V)} \le C_{s,U_1,U_2} \left(\|P_\varepsilon u\|_{L^2_s(U_2, p^*\omega_V)} + \|u\|_{L^2(U_2, p^*\omega_V)}\right)$$

for all $u \in \mathcal{C}_c^\infty(U_2)$. Let $\mathcal{U} = (U_i)_{i\in\mathbb{N}}$ be a relatively compact exhaustion of $\widehat{V}^{\mathrm{reg}} \setminus \mathrm{Exc}(p)$. Differentiating (3.2.11) on both sides and using a bootstrapping argument, we obtain $\|\rho_\varepsilon\|_{L^2_s(U_1, p^*\omega_V)} < C(s, \mathcal{U})$ where $C(s, \mathcal{U})$ does not depend on $\varepsilon \in (0, 1]$. By Rellich's theorem, there exists a subsequence $(\rho_{\varepsilon_j})_{j\in\mathbb{N}}$ such that ρ_{ε_j} converges to ρ in $\mathcal{C}^l(\overline{U_1})$ for all $l \in \mathbb{N}$ when $\varepsilon_j \to 0$. Therefore $\mathrm{dd^c}(\rho p^*\omega_V^{n-1}) = \lim_{j\to+\infty} \mathrm{dd^c}(\rho_{\varepsilon_j}\omega_{\varepsilon_j}^{n-1}) = 0$ on U_1. Using a diagonal argument, we infer that there is a smooth function ρ on $\widehat{V}^{\mathrm{reg}} \setminus \mathrm{Exc}(p) \xrightarrow[p]{\sim} V^{\mathrm{reg}}$ which is bounded between 1 and C_G, and satisfies $\mathrm{dd^c}(\rho\omega_V^{n-1}) = 0$ on V^{reg}. □

Recall that $\mu : \widetilde{X} \to X$ is a resolution of singularities given by a composition of finitely many blowups with smooth centers,

$$\widetilde{X} = X_m \xrightarrow{p_m} X_{m-1} \xrightarrow{p_{m-1}} \cdots \xrightarrow{p_2} X_1 \xrightarrow{p_1} X_0 = X$$

where $p_k : X_k \to X_{k-1}$ is a blowup with non-singular center. By Gauduchon's theorem [Gau77], for each $\varepsilon(m) = (\varepsilon_1, \cdots, \varepsilon_m) \in (0,1]^m$, there exists a normalized Gauduchon factor $\rho_{\varepsilon(m)}$ on X_m with respect to the metric $\omega_{\varepsilon(m)}$. Proposition 3.2.3 says that for any fixed $(\varepsilon_1, \cdots, \varepsilon_{m-1}) \in (0,1]^{m-1}$ there exists a normalized bounded Gauduchon factor $\rho_{(\varepsilon_1,\cdots,\varepsilon_{m-1})}$ on X_{m-1}^{reg} with respect to $\omega_{(\varepsilon_1,\cdots,\varepsilon_{m-1})}$. Again fixing $(\varepsilon_1, \cdots, \varepsilon_{m-2}) \in (0,1]^{m-2}$ and taking the limit $\varepsilon_{m-1} \to 0$, one gets a normalized bounded Gauduchon factor $\rho_{(\varepsilon_1,\cdots,\varepsilon_{m-2})}$ on X_{m-2}^{reg} with respect to the metric $\omega_{(\varepsilon_1,\cdots,\varepsilon_{m-2})}$. Repeating the argument, we obtain a bounded Gauduchon factor ρ on X^{reg} with respect to ω. The uniqueness of ρ has already been given in [Pan22, Theorem B]. This completes the proof of Theorem 3.0.1.

3.3 Families of singular Gauduchon metrics

In this section, we study families of hermitian varieties and establish a uniform upper bound on the normalized bounded Gauduchon factor on each fibre. We then use the uniform estimate of normalized Gauduchon factors to prove the Sup-L^1 comparison of quasi-plurisubharmonic functions in families of hermitian varieties where all fibres have only isolated singularities.

Setup (F). *Let $(\mathcal{X}, \omega)$ be an $(n+1)$-dimensional hermitian variety. Suppose that $\pi : \mathcal{X} \to \mathbb{D}$ is a proper, surjective, holomorphic map with connected fibres and for all $t \in \mathbb{D}$, the fibre $X_t := \pi^{-1}(t)$ is a variety. For each $t \in \mathbb{D}$, we define a hermitian metric ω_t on the fibre X_t by restriction (i.e. $\omega_t := \omega|_{X_t}$).*

3.3.1 Uniform estimate of normalized Gauduchon factors in families

By Theorem 3.0.1, one can find a normalized bounded Gauduchon factor ρ_t on each fibre (X_t, ω_t). Using Theorem 3.1.1, we obtain a uniform estimate in families:

Proposition 3.3.1 (=Proposition 3.0.2)**.** *In Setting (F), there is a uniform upper bound C_G of the normalized Gauduchon factors ρ_t on X_t for all $t \in \overline{\mathbb{D}}_{1/2}$.*

Sketch of proof of Proposition 3.3.1. To apply Theorem 3.1.1, we need to figure out that the constants $B(\omega_t)$, $C_S(\omega_t)$, $C_P(\omega_t)$, $\mathrm{Vol}(X_t, \omega_t)$ are uniformly bounded.

In a local embedding, we have $-B\omega^n \leq \mathrm{dd^c}\omega^{n-1} \leq B\omega^n$ on $\mathcal{X}$; hence there is a constant $B \geq 0$ independent of t such that

$$-B\omega_t^n \leq \mathrm{dd}_t^{\mathrm{c}}\omega_t^{n-1} \leq B\omega_t^n \tag{3.3.1}$$

for all $t \in \overline{\mathbb{D}}_{1/2}$. By the continuity of the total mass of currents $(\omega^n \wedge [X_t])_{t \in \overline{\mathbb{D}}_{1/2}}$ (cf. [Pan22, Section 1.4] for details), we have a uniform constant $C_V \geq 1$ such that

$$C_V^{-1} \leq \mathrm{Vol}_{\omega_t}(X_t) \leq C_V, \quad \forall t \in \overline{\mathbb{D}}_{1/2}. \tag{3.3.2}$$

A uniform Sobolev constant in family can be produced by Wirtinger inequality and Michael–Simon's Sobolev inequality on minimal submanifolds [MS73]. A uniform Poincaré constant in families was given by Yoshikawa [Yos97] and Ruan–Zhang [RZ11]. For convenience, the reader is also referred to [DGG20, Proposition 3.8 and 3.10] (the Kähler assumption made by Di Nezza–Guedj–Guenancia being unnecessary for this part of argument). Therefore, for all $t \in \overline{\mathbb{D}}_{1/2}$, there exists uniform constants C_S and C_P such that

$$\forall f \in L_1^2(X_t^{\mathrm{reg}}), \quad \left(\int_{X_t} |f|^{\frac{2n}{n-1}} \omega_t^n\right)^{\frac{n-1}{n}} \leq C_S \left(\int_{X_t} |\mathrm{d}f|_{\omega_t}^2 \omega_t^n + \int_{X_t} |f|^2 \omega_t^n\right) \tag{3.3.3}$$

and

$$\forall f \in L_1^2(X_t^{\mathrm{reg}}) \text{ and } \int_{X_t} f\omega_t^n = 0, \quad \int_{X_t} |f|^2 \omega_t^n \leq C_P \int_{X_t} |\mathrm{d}f|_{\omega_t}^2 \omega_t^n. \tag{3.3.4}$$

Combining Theorem 3.1.1, (3.3.1), (3.3.2), (3.3.3) and (3.3.4), this completes the proof of Corollary 3.3.1. □

3.3.2 Applications in families

In this section, we establish a uniform Sup-L^1 comparison in the case where all fibres X_t, $t \in \mathbb{D}$ have only isolated singularities.

Bounded integral of Laplacian

We first recall a useful extension property of the singular Gauduchon metrics. Following the proof of [Pan22, Theorem B], one can establish the following lemma:

Lemma 3.3.2. *Let (X, ω) be a compact hermitian variety, ρ be a normalized bounded Gauduchon factor with respect to ω, and $p : \widetilde{X} \to X$ be a resolution of singularities which consists of finitely many blowups. Then the $(n-1, n-1)$-form $p^*\left(\rho\omega^{n-1}\right)$ on $\widetilde{X} \setminus \mathrm{Exc}(p)$ extends to a pluriclosed positive current on $\widetilde{X}$.*

Sketch of proof of Lemma 3.3.2. Let $\widetilde{\omega}$ be a hermitian metric on $\widetilde{X}$ such that $\widetilde{\omega} \geq p^*\omega$. Following the construction in [CGP13, Section 9], one obtains good cut-off functions $(\widetilde{\chi}_\delta)_{\delta>0} \subset \mathcal{C}_c^\infty(\widetilde{X} \setminus \mathrm{Exc}(p))$ as in Lemma 3.1.2 such that

- $\widetilde{\chi}_\delta$ is increasing to the characteristic function of $\widetilde{X} \setminus \mathrm{Exc}(p)$ when δ decreases to 0;
- $\int_{\widetilde{X}} |\mathrm{dd^c}\widetilde{\chi}_\delta \wedge \widetilde{\omega}^{n-1}| \to 0$ when $\delta \to 0$;
- $\int_{\widetilde{X}} \mathrm{d}\widetilde{\chi}_\delta \wedge \mathrm{d^c}\chi_\delta \wedge \widetilde{\omega}^{n-1} \to 0$ when $\delta \to 0$.

Then one can finish the proof by same arguments as in [Pan22, page 13-14]. □

Next, we prove that the integral of the Laplacian of ω_t-psh functions are uniformly bounded:

Lemma 3.3.3. *Under Setting (F), there exists a constant $C_{\mathrm{Lap}} > 0$ such that for all $t \in \overline{\mathbb{D}}_{1/2}$ and for all $\varphi_t \in \mathrm{PSH}(X_t, \omega_t)$,*

$$\int_{X_t} (\omega_t + \mathrm{dd}_t^c \varphi_t) \wedge \omega_t^{n-1} \leq C_{\mathrm{Lap}}.$$

Proof. Let $p : \mathcal{Y} \to \mathcal{X}$ be a resolution of singularities of $\mathcal{X}$. Pick $\widetilde{\omega}$ a hermitian metric on $\mathcal{Y}$ such that $\widetilde{\omega} \geq p^*\omega$. The induced map $\mu := \pi \circ p : \mathcal{Y} \to \mathbb{D}$ is surjective. By generic smoothness, it is smooth over a complement of proper analytic subset $A \subset \mathbb{D}$. In particular, A is discrete so we can assume that $Y_t := \mu^{-1}(t)$ is smooth for all $t \neq 0$ up to shrinking $\mathbb{D}$. We denote by p_t the restriction $p|_{Y_t} : Y_t \to X_t$ of p to the fibre Y_t. For each $t \neq 0$, the map $p_t : Y_t \to X_t$ is bimeromorphic, i.e. it is a resolution of singularities of X_t.

Now, we fix $t \neq 0$. For all $\varphi_t \in \mathrm{PSH}(X_t, \omega_t)$, the pullback function $p_t^*\varphi_t$ is $p_t^*\omega_t$-psh and thus $p_t^*\varphi_t \in \mathrm{PSH}(Y_t, \widetilde{\omega}_t)$. By [BK07, Theorem 1], there is a sequence of smooth $\widetilde{\omega}_t$-psh functions $(\psi_{t,j})_j$ decreasing to $p_t^*\varphi_t$. Then we have

$$\int_{Y_t} (\widetilde{\omega}_t + dd^c_t \psi_{t,j}) \wedge p_t^*\omega_t^{n-1} \xrightarrow[j\to+\infty]{} \int_{Y_t} (\widetilde{\omega}_t + dd^c_t p_t^*\varphi_t) \wedge p_t^*\omega_t^{n-1}.$$

Let $\rho_t \in \mathcal{C}^\infty(X_t^{\mathrm{reg}}) \cap L^\infty(X_t^{\mathrm{reg}})$ be the normalized bounded Gauduchon factor with respect to ω_t. Applying Lemma 3.3.2 and Proposition 3.3.1, one can derive

$$\begin{aligned}\int_{Y_t} (\widetilde{\omega}_t + dd^c_t \psi_{t,j}) \wedge p_t^*\omega_t^{n-1} &\leq \int_{Y_t} (\widetilde{\omega}_t + dd^c_t \psi_{t,j}) \wedge p_t^*(\rho_t\omega_t^{n-1}) \\ &= \int_{Y_t} \widetilde{\omega}_t \wedge p_t^*(\rho_t\omega_t^{n-1}) \leq \int_{Y_t} C_G \widetilde{\omega}_t \wedge p_t^*\omega_t^{n-1} \leq C_G \mathrm{Vol}_{\widetilde{\omega}_t}(Y_t).\end{aligned}$$

Therefore, one finds for all $t \in \overline{\mathbb{D}}_{1/2} \setminus \{0\}$

$$\int_{X_t} (\omega_t + dd^c_t \varphi_t) \wedge \omega_t^{n-1} \leq C_G \mathrm{Vol}_{\widetilde{\omega}_t}(Y_t).$$

Similarly to (3.3.2), there is a uniform bound $C_{V,Y} > 0$ of $\mathrm{Vol}_{\widetilde{\omega}_t}(Y_t)$ for all $t \in \overline{\mathbb{D}}_{1/2}$.

When $t = 0$, standard arguments (cf eg. [GZ17, Proposition 8.5]) allow one to find a constant $C > 0$ such that $-C \leq \int_{X_0} (\varphi_0 - \sup_{X_0} \varphi_0)\omega_0^n$ for all $\varphi_0 \in \mathrm{PSH}(X_0, \omega_0)$. One can prove that there exists a constant $C' > 0$ such that for all $\varphi_0 \in \mathrm{PSH}(X_0, \omega_0)$,

$$\int_{X_0} (\omega_0 + dd^c_0 \varphi_0) \wedge \omega_0^{n-1} \leq C'$$

and we conclude the proof. □

Sup-L^1 comparison

Finally, we establish a Sup-L^1 comparison of ω_t-psh functions in families of hermitian varieties with isolated singularities. This partially answers the conjecture [DGG20, Conjecture 3.1] and generalizes [Pan23, Proposition 5.2] which showed that Sup-L^1 comparison holds for *smoothing* families with only isolated singularities.

Proposition 3.3.4 (=Proposition 3.0.3)**.** *Under Setting (F), suppose that each fibre X_t has only isolated singularities. Then there is a constant $C_{SL} > 0$ such that for all $t \in \overline{\mathbb{D}}_{1/2}$*

$$\forall \varphi_t \in \mathrm{PSH}(X_t, \omega_t) \quad \sup_{X_t} \varphi_t - C_{SL} \leq \frac{1}{V_t} \int_{X_t} \varphi_t \omega_t^n \leq \sup_{X_t} \varphi_t$$

where V_t is the volume of X_t with respect to ω_t.

We give a proof of Proposition 3.3.4 closely following [DGG20, Section 3.3]:

Sketch of proof of Proposition 3.3.4. Note that we only need to take care of sequences of sup-normalized ω_{t_k}-psh functions $(\varphi_{t_k})_k$ where $t_k \xrightarrow[k\to+\infty]{} 0$.

Step 1: Choose a good covering and a test function. Let $\mathcal{Z}$ be the singular locus of π. Following the same argument as in [DGG20, page 30, Step 2], up to shrinking $\mathbb{D}$, we can find a finite open covering $(\mathcal{V}_i)_{i\in I}$ of $\mathcal{X}$ such that

(i) each point of $\mathcal{Z}\cap X_0$ belongs to exactly one element of $\mathcal{V}_i$ of the covering, we denote by J the collection of indices of these open subsets;

(ii) on each $\mathcal{V}_i$, we have a smooth strictly psh function g_i such that

$$C_g^{-1}\mathrm{dd^c}g_i \le \omega \le C_g\mathrm{dd^c}g_i \quad \text{and} \quad 0 \le g_i \le C_g$$

for a uniform constant $C_g > 0$;

(iii) for all $i\in I$, $\mathcal{Z}\cap\partial\mathcal{V}_i = \emptyset$;

(iv) for each $i\in J$, there is a relatively compact open subset $\mathcal{W}_i \Subset \mathcal{V}_i$ with $\mathcal{Z}\cap\mathcal{W}_i \neq \emptyset$, $\mathcal{Z}\cap\partial\mathcal{W}_i = \emptyset$ and $\mathcal{Z}\cap\partial\mathcal{W} = \emptyset$ where $\mathcal{W} := \cup_{i\in J}\mathcal{W}_i$.

In particular, up to shrinking $\mathbb{D}$, one can find $\delta > 0$ such that for any $t\in\mathbb{D}$,

$$\mathrm{dist}_\omega(\partial\mathcal{W}\cap X_t, \mathcal{Z}) \ge \delta.$$

Let χ_i be a cut-off function supported in $\mathcal{V}_i$ and $\chi_i \equiv 1$ in a neighborhood of $\mathcal{W}_i$. Set $g = \sum_{i\in J}\chi_i g_i$. Obviously, we have $\omega \le C_g\mathrm{dd^c}g$ on $\mathcal{W}$. Furthermore, we may assume $-C_g\omega \le \mathrm{dd^c}g \le C_g\omega$ on $\mathcal{X}$ by choosing larger C_g.

Step 2: Uniform L^1-estimate away from singularities. Define a set

$$\mathcal{R} := \{p\in\mathcal{X} \mid \mathrm{dist}_\omega(p,\mathcal{Z}) > \delta/2\}.$$

Since $\mathcal{R}^c$ lies in $\mathcal{W}$, after shrinking $\mathbb{D}$, we can cover $\mathcal{R}$ by finitely many open subsets $(\mathcal{U}_i)_i$ away from the singular locus. We may assume that π is locally trivial on $\mathcal{R}$ with respect to $(\mathcal{U}_i)_i$ because π is a submersion on $\mathcal{R}$.

Following the argument in [DGG20, page 31, Step 3], one can prove that there is a constant $C > 0$ and a subsequence of $(t_k)_k$ such that

$$\sup_{\mathcal{R}\cap X_{t_k}} \varphi_{t_k} \ge -C.$$

Then one can apply the similar method in locally trivial cases (cf. [DGG20, Section 3.2]) to show that

$$\int_{\mathcal{R}\cap X_{t_k}} (-\varphi_{t_k})\omega_{t_k}^n \le C_{\mathcal{R}}$$

for some uniform constant $C_{\mathcal{R}} > 0$.

Step 3: Conclusion. Recall that on $\mathcal{W}$ we have $\omega \leq C_g \mathrm{dd^c} g$. Define a smooth (n,n)-form $\Omega := \omega^n - C_g^n (\mathrm{dd^c} g)^n$. It is easy to see that $\Omega|_{\mathcal{W}\cap X_t} \leq 0$ and $\Omega_t := \Omega|_{X_t} \leq C_\Omega \omega_t^n$ for some uniform constant $C_\Omega > 0$. Note that $\mathcal{R}^c \subset \mathcal{W}$. We have

$$\begin{aligned}\int_{X_{t_k}} (-\varphi_{t_k})\Omega_{t_k} &= \int_{\mathcal{R}\cap X_{t_k}} (-\varphi_{t_k})\Omega_{t_k} + \int_{\mathcal{R}^c\cap X_{t_k}} (-\varphi_{t_k})\Omega_{t_k} \\ &\leq C_\Omega \int_{\mathcal{R}\cap X_{t_k}} (-\varphi_{t_k})\omega_{t_k}^n \leq C_{\mathcal{R}} C_\Omega.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\int_{X_{t_k}} (-\varphi_{t_k})(\mathrm{dd}^c_{t_k} g)^n &= \int_{X_{t_k}} -g \mathrm{dd}^c_{t_k} \varphi_{t_k} \wedge (\mathrm{dd}^c_{t_k} g)^{n-1} \\ &= -\int_{X_{t_k}} g(\omega_{t_k} + \mathrm{dd}^c_{t_k}\varphi_{t_k}) \wedge (\mathrm{dd}^c_{t_k} g)^{n-1} + \int_{X_{t_k}} g\omega_{t_k} \wedge (\mathrm{dd}^c_{t_k} g)^{n-1} \\ &\leq C_g^n \int_{X_{t_k}} (\omega_{t_k} + \mathrm{dd}^c_{t_k}\varphi_{t_k}) \wedge \omega_{t_k}^{n-1} + C_g^n \operatorname{Vol}_{\omega_t}(X_t) \\ &\leq C_g^n C_{\mathrm{Lap}} + C_g^n \operatorname{Vol}_{\omega_t}(X_t).\end{aligned}$$

The fourth line comes directly from Lemma 3.3.3. All in all, we obtain a uniform L^1-estimate of sup-normalized ω_{t_k}-psh functions:

$$\begin{aligned}\int_{X_{t_k}} (-\varphi_{t_k})\omega_{t_k}^n &= C_g^n \int_{X_{t_k}} (-\varphi_{t_k})(\mathrm{dd}^c_{t_k} g)^n + \int_{X_{t_k}} (-\varphi_{t_k})\Omega_{t_k} \\ &\leq C_g^{2n} C_{\mathrm{Lap}} + C_g^{2n} C_V + C_{\mathcal{R}} C_\Omega.\end{aligned}$$

□

3.4 Uniform Sobolev inequality: Ou's partial result

In this section, we recall Ou's partial solution [Ou22, Prop. 4.10] to Conjecture (SC). Fix $k \in [0, 2n)$ and we refine a few estimates in the original proof. In the sequel, we denote by $\mathcal{H}^k$ the k-dimensional Hausdorff measure.

We first recall a corollary of Stokes formula and Wirtinger inequality (cf. [Dem12, p. 137]):

Lemma 3.4.1. *Let (X,ω) be a Kähler manifold and $Y \subset X$ be a complex submanifold. Then Y is a minimal submanifold with respect to ω.*

Lemma 3.4.2 ([Chi89, Ch 3, Sec. 15, Prop. 1 and 2])**.** *Let A be a pure n-dimensional analytic set of the ball $\mathbb{B}_R$ in $\mathbb{C}^N$ containing the origin. The function $r \mapsto r^{-2n}\mathcal{H}^{2n}(A \cap \mathbb{B}_r)$ decreases as $r \to 0$ and the limit*

$$m(A,0) = \lim_{r\to 0} \frac{\mathcal{H}^{2n}(A\cap \mathbb{B}_r)}{\pi^n r^{2n}/n!}$$

is the multiplicity of A at 0.

Lemma 3.4.3 ([Ou22, Lem. 4.12]). *Let $X \subset \mathbb{C}^N$ be an n-dimensional analytic subvariety defined around the origin $0 \in \mathbb{C}^N$. Then for all $k \in [0, 2n)$, there is a constant $C_1(n,k,D) := \frac{2nD}{2n-k}$ such that for all $\rho > 0$,*

$$\int_{B_\rho(0)\cap X} |z|^{-k} \mathrm{d}\mathcal{H}^{2n} \leq C_1 \rho^{2n-k}$$

where D is a constant so that $\mathcal{H}^{2n}(B_r(0) \cap X) \leq Dr^{2n}$ for all $r > 0$.

Proof. We indicate a slightly simpler variant of Ou's proof. By Lebesgue integral formula, for all non-negative μ-measurable function f, we have

$$\int_E f \mathrm{d}\mu = \int_0^{+\infty} \mu\left(\{f > s\} \cap E\right) \mathrm{d}s.$$

From the above formula, we derive

$$\begin{aligned}
\int_{B_\rho(0)\cap X} |z|^{-k} \mathrm{d}\mathcal{H}^{2n} &= \int_0^{+\infty} \mathcal{H}^{2n}\left(\{|z|^{-k} > s\} \cap (B_\rho(0) \cap X)\right) \mathrm{d}s \\
&= \int_0^{+\infty} \mathcal{H}^{2n}\left(\{|z| < s^{-1/k}\} \cap (B_\rho(0) \cap X)\right) \mathrm{d}s \\
&= \int_{\rho^{-k}}^{+\infty} \mathcal{H}^{2n}\left(B_{s^{-1/k}}(0) \cap X\right) \mathrm{d}s + \int_0^{\rho^{-k}} \mathcal{H}^{2n}(B_\rho(0) \cap X) \mathrm{d}s \\
&\leq \int_0^{\rho} k t^{-k-1} \mathcal{H}^{2n}\left(B_t(0) \cap X\right) \mathrm{d}t + D\rho^{2n-k}.
\end{aligned}$$

Thus, we obtain

$$\int_{B_\rho(0)\cap X} |z|^{-k} \mathrm{d}\mathcal{H}^{2n} \leq \frac{2nD}{2n-k} \rho^{2n-k}.$$

□

From here on, the proof is identical to Ou's original one [Ou22, Sec. 4.C]. We give more details for the reader's convenience.

Lemma 3.4.4 ([Ou22, Lem. 4.14]). *For any $A > 0$, there is a constant $\zeta > 0$ such that for all $\varepsilon \in (0, 1]$,*

$$\int_{B_{\varepsilon\zeta}(0)} \omega_\varepsilon^n \leq A\varepsilon^{2n}.$$

Proof. The proof is extracted from [Ou22, Lem. 4.14]. Observe that

$$\begin{aligned}
\omega_\varepsilon^n|_X &= \left(\mathrm{dd^c}|z|^2 + \varepsilon^2 \mathrm{dd^c} \log|z|^2\right)^n\Big|_X = \left((1-\varepsilon^2)\omega_0 + \varepsilon^2\omega_1\right)^n\Big|_X \\
&= \left(\sum_{j=0}^{n-1} \binom{n}{j} (1-\varepsilon^2)^{n-j} \varepsilon^{2j} \omega_0^{n-j} \wedge \omega_1^j\right) + \varepsilon^{2n}\, \omega_1^n|_X\,.
\end{aligned}$$

Note that
$$\omega_0 \le \omega_1 \le \frac{B}{|z|^2}\omega_0.$$
Hence, we have
$$\omega_\varepsilon^n|_X \le \left(\sum_{j=0}^{n-1}\binom{n}{j}\frac{A^j\varepsilon^{2j}}{|z|^{2j}}\,\omega_0^n|_X\right) + \varepsilon^{2n}\,\omega_1^n|_X\,.$$
By Lemma 3.4.3,
$$\int_{B_{\varepsilon\rho}(0)\cap X}\left(\sum_{j=0}^{n-1}\binom{n}{j}\frac{B^j\varepsilon^{2j}}{|z|^{2j}}\right)\omega_0^n \le \sum_{j=0}^{n-1}A^j\varepsilon^{2j}C_1(n,2j,D)(\varepsilon\rho)^{2(n-j)} \le C_2(n,D,B)\varepsilon^{2n}\rho^2.$$
On the other hand, one can identify $(B_{\varepsilon\rho}(0)\setminus\{0\},\omega_1)$ to be a $\varepsilon\rho$-tubular neighborhood of the exceptional divisor E of the blowup at the origin $\widehat{\mathbb{C}}^N \to \mathbb{C}^N$. One can find a constant C_3 such that
$$\int_{B_{\varepsilon\rho}(0)\cap X}\omega_1^n \le C_3(\varepsilon\rho)^2$$
and thus, we obtain
$$\int_{B_{\varepsilon\rho}(0)\cap X}\omega_\varepsilon^n \le C_2\varepsilon^{2n}\rho^2 + C_3\varepsilon^{2n+2}\rho^2.$$
□

We recall a crucial result from Whitney's stratification theorem [Whi65, Thm. 19.2]:

Lemma 3.4.5 ([Ou22, Lem. 4.15]). *There is an open neighborhood U of 0 in $\mathbb{C}^N$ and a constant $\gamma > 0$ such that for any point $x \neq 0$ of $X^{\mathrm{reg}} \cap U$, the following inequality holds:*
$$\left\langle e_x, e_x^T\right\rangle = |e_x^T|^2 \ge \gamma,$$
where e_x is the unit vector pointing from 0 to x and e_x^T is the orthogonal projection of e_x on to T_xX^{reg}.

Lemma 3.4.6 ([Ou22, Lem. 4.16]). *With the open neighborhood U in the above lemma, for any $\rho > 0$ and for any non-negative C^1-function h supported in the smooth locus of $U \cap X \setminus \{0\}$, we have*
$$\int_{X\cap\partial B_\rho(0)} h\,\mathrm{d}\sigma_0 \le \frac{1}{\gamma^{1/2}}\int_{X\setminus B_\rho(0)}|\mathrm{d}h|_{\omega_0}\,\omega_0^n$$
where $\mathrm{d}\sigma_0$ is the measure induced by ω_0.

Proof. Set $r(x) := |x|$. We have $|\nabla^X r| = |(\nabla^{\mathbb{C}^N} r)^T| \le |\nabla^{\mathbb{C}^N} r| = 1$ and $\mathrm{div}_X(r\nabla^{\mathbb{C}^n} r) = 2n$ (cf. [HS74, Eq. (3.6)]). Note that for any smooth function f and for any vector field $V : X^{\mathrm{reg}} \to T\mathbb{C}^N$, we have
$$\mathrm{div}_X(fV) = f\,\mathrm{div}_X(V) + \left\langle V, \nabla^X f\right\rangle = f\,\mathrm{div}_X(V) + \left\langle V^T, \nabla^X f\right\rangle;$$
hence
$$\mathrm{div}_X(\nabla^{\mathbb{C}^N} r) = \frac{1}{r}\left(\mathrm{div}_X(\nabla^{\mathbb{C}^N} r) - \left|\nabla^X r\right|^2\right) \ge 0.$$

From [HS74, Eq. (3.2)], we obtain a divergence formula

$$\int_{X\setminus B_\rho(0)} \mathrm{div}_X(V)\mathrm{d}V_X = -\int_{X\setminus B_\rho(0)} \langle V, H_X\rangle \,\mathrm{d}V_X + \int_{X\cap\partial B_\rho(0)} \langle V, \nu\rangle \,\mathrm{d}V_{\partial B_\rho(0)\cap X},$$

where H_X is the mean curvature of X and ν it the exterior normal vector field to $X\setminus B_\rho(0)$ on $X\cap\partial B_\rho(0)$ (in this case $\nu = -\frac{\nabla^X r}{|\nabla^X r|}$). By Wirtinger inequality, we know that $H_X \equiv 0$. Then we get

$$\begin{aligned} 0 \le \int_{X\setminus B_\rho(0)} h\,\mathrm{div}_X(\nabla^{\mathbb{C}^N} r)\mathrm{d}V_X &= \int_{X\setminus B_\rho(0)} \mathrm{div}_X(h\nabla^{\mathbb{C}^N} r) - \left\langle \nabla^{\mathbb{C}^N} r, \nabla^X h\right\rangle \mathrm{d}V_X \\ = -\int_{X\cap\partial B_\rho(0)} h\left|\nabla^X r\right| \mathrm{d}V_{X\cap\partial B_\rho(0)} &- \int_{X\setminus B_\rho(0)} \left\langle \nabla^{\mathbb{C}^N} r, \nabla^X h\right\rangle \mathrm{d}V_X. \end{aligned}$$

Thus,

$$\begin{aligned} \int_{X\cap\partial B_\rho(0)} h\left|\nabla^X r\right| \mathrm{d}V_{\partial B_\rho(0)\cap X} &\le \int_{X\setminus B_\rho(0)} \left|\left\langle \nabla^{\mathbb{C}^N} r, \nabla^X h\right\rangle\right| \mathrm{d}V_X \\ &\le \int_{X\setminus B_\rho(0)} \left|\nabla^{\mathbb{C}^N} r\right| \left|\nabla^X h\right| \mathrm{d}V_X \\ &= \int_{X\setminus B_\rho(0)} \left|\nabla^X h\right| \mathrm{d}V_X. \end{aligned}$$

By Lemma 3.4.5, we have $|\nabla^X r| \ge \gamma^{1/2}$; hence the result follows. □

Proposition 3.4.7 ([Ou22, Prop. 4.10]). *Let $X \subset \mathbb{C}^N$ be an n-dimensional complex analytic subvariety defined around the origin 0. Let z be a coordinate system of $\mathbb{C}^N$. We consider the Kähler forms for $0 < \varepsilon \le 1$,*

$$\omega_\varepsilon = \mathrm{dd^c}(|z|^2 + \varepsilon^2 \log|z|^2).$$

Then there is a neighborhood U of 0 in $\mathbb{C}^N$ and a uniform Sobolev constant C_S such that for all $\varepsilon \in (0,1]$, for any non-negative $\mathcal{C}^1$-function h which is compactly supported in $U \cap X^{\mathrm{reg}} \setminus \{0\}$,

$$\int_X h\omega_\varepsilon^n \le C_S \int_X |\mathrm{d}h|_{\omega_\varepsilon} \omega_\varepsilon^n.$$

Proof. Let U be an open set as in Lemma 3.4.5. Note that one has an isometry as follows:

$$\begin{array}{ccc} (\widehat{\mathbb{C}}, \omega_\varepsilon) & \xrightarrow{\simeq} & (\widehat{\mathbb{C}}, \varepsilon^2\omega_1) \\ \Psi & & \Psi \\ z & \longmapsto & z' := z/\varepsilon \end{array}.$$

Then one can find constants $b, R > 0$ such that on $(\widehat{\mathbb{C}}^N, \omega_\varepsilon)$

- the sectional curvature of is bounded from above by $\frac{b^2}{\varepsilon^2}$;
- the injectivity radius is at least δR.

Let A be a small positive constant such that the following inequality holds

$$b^{-1}\arcsin\left(b\left(\frac{2A}{v_{2n}}\right)^{1/2n}\right)\leq\frac{R}{2}$$

where v_{2n} is the volume of unit ball in $\mathbb{R}^{2n}$. From Lemma 3.4.4, we find $\zeta>0$ such that for all $\varepsilon\in(0,1)$

$$\int_{B_{\varepsilon\zeta}(0)\cap X}\omega_\varepsilon^n\leq A\varepsilon^{2n}.$$

Then by [HS74, Theroem 2.1] and Lemma 3.4.1, there is a constant $c>0$ such that for any non-negative $\mathcal{C}^1$ function g supported in $X^{\mathrm{reg}}\cap B_{\varepsilon\zeta}(0)\setminus\{0\}$, we have

$$\left(\int_{X\cap B_{\delta\zeta}(0)}g^{\frac{2n}{2n-1}}\omega_\varepsilon^n\right)^{\frac{2n-1}{2n}}\leq c\int_{X\cap B_{\delta\zeta}(0)}|\mathrm{d}g|_{\omega_\varepsilon}\omega_\varepsilon^n. \tag{3.4.1}$$

Let h be a non-negative $\mathcal{C}^1$ function supported in $X^{\mathrm{reg}}\cap U\setminus\{0\}$. Multiplying smooth cut-offs approximating $\mathbb{1}_{B_{\varepsilon\zeta}(0)}$, from 3.4.1, we obtain

$$\left(\int_{X\cap B_{\varepsilon\zeta}(0)}h^{\frac{2n}{2n-1}}\omega_\varepsilon^n\right)^{\frac{2n-1}{2n}}\leq c\left(\int_{X\cap B_{\varepsilon\zeta}(0)}|\mathrm{d}h|_{\omega_\varepsilon}\omega_\varepsilon^n+\int_{X\cap\partial B_{\varepsilon\zeta}(0)}h\mathrm{d}\sigma_\varepsilon\right)$$

where $\mathrm{d}\sigma_\varepsilon$ is the measure on $\partial B_{\varepsilon\zeta}(0)$ induced by ω_ε.

Recall that $\omega_0\leq\omega_1\leq\frac{B}{|z|^2}\omega_0$. On $\{|z|\geq\varepsilon\zeta\}$, we have $\omega_0\leq\omega_\varepsilon\leq\left(1+\frac{B}{\zeta^2}\right)\omega_0$. Again, multiplying h by smooth cut-offs approximating $\mathbb{1}_{\mathbb{C}^N\setminus B_{\varepsilon\zeta}(0)}$ and applying Michael–Simon's Sobolev inequality, one finds a uniform constant $c'=c'(n,B,\zeta)$ and

$$\left(\int_{X\setminus B_{\varepsilon\zeta}(0)}h^{\frac{2n}{2n-1}}\omega_\varepsilon^n\right)^{\frac{2n-1}{2n}}\leq c'\left(\int_{X\setminus B_{\varepsilon\zeta}(0)}|\mathrm{d}h|_{\omega_\varepsilon}\omega_\varepsilon^n+\int_{X\cap\partial B_{\varepsilon\zeta}(0)}h\mathrm{d}\sigma_\varepsilon\right).$$

Note that $\mathrm{d}\sigma_\delta\leq\left(1+\frac{B}{\zeta^2}\right)^{2n-1}\mathrm{d}\sigma_0$. From Lemma 3.4.6, we get

$$\int_{X\cap\partial B_{\varepsilon\zeta}(0)}h\mathrm{d}\sigma_0\leq\frac{1}{\gamma^{1/2}}\int_{X\setminus B_{\varepsilon\zeta}(0)}|\mathrm{d}h|_{\omega_0}\omega_0^n\leq\frac{\left(1+\frac{B}{\zeta^2}\right)}{\gamma^{1/2}}\int_{X\setminus B_{\varepsilon\zeta}(0)}|\mathrm{d}h|_{\omega_\varepsilon}\omega_0^n\leq\frac{\left(1+\frac{B}{\zeta^2}\right)}{\gamma^{1/2}}\int_{X\setminus B_{\varepsilon\zeta}(0)}|\mathrm{d}h|_{\omega_\varepsilon}\omega_\varepsilon^n.$$

Thus, we have

$$\left(\int_{X\cap B_{\varepsilon\zeta}(0)}h^{\frac{2n}{2n-1}}\omega_\varepsilon^n\right)^{\frac{2n-1}{2n}}\leq c\left(1+\frac{\left(1+\frac{B}{\zeta^2}\right)^{2n}}{\gamma^{1/2}}\right)\int_{X\cap B_{\varepsilon\zeta}(0)}|\mathrm{d}h|_{\omega_\varepsilon}\omega_\varepsilon^n.$$

and

$$\left(\int_{X\setminus B_{\varepsilon\zeta}(0)} h^{\frac{2n}{2n-1}}\omega_\varepsilon^n\right)^{\frac{2n-1}{2n}} \leq c'\left(1+\frac{\left(1+\frac{B}{\zeta^2}\right)^{2n}}{\gamma^{1/2}}\right)\int_{X\setminus B_{\varepsilon\zeta}(0)} |\mathrm{d}h|_{\omega_\varepsilon}\,\omega_\varepsilon^n.$$

All in all, we conclude that

$$\left(\int_X h^{\frac{2n}{2n-1}}\omega_\varepsilon^n\right)^{\frac{2n-1}{2n}} \leq (c+c')\left(1+\frac{\left(1+\frac{B}{\zeta^2}\right)^{2n}}{\gamma^{1/2}}\right)\int_X |\mathrm{d}h|_{\omega_\varepsilon}\,\omega_\varepsilon^n.$$

□

Chapter 4

Families of singular Chern–Ricci flat metrics

In this chapter, we provide (with slight modification) the content of the article [Pan23].

We prove uniform a priori estimates for degenerate complex Monge–Ampère equations on a family of hermitian varieties. This generalizes a theorem of Di Nezza–Guedj–Guenancia to hermitian contexts. The main result can be applied to study the uniform boundedness of Chern–Ricci flat potentials in conifold transitions.

Introduction

Let $\pi : \mathcal{X} \to \mathbb{D}$ be a family of hermitian varieties (i.e. irreducible, reduced, complex analytic spaces). Recently, Di Nezza–Guedj–Guenancia [DGG20] developed the first steps of a pluripotential theory in families of Kähler spaces and proved uniform bounds for Kähler–Einstein potentials in several cases. The main purpose of this article is to generalize their theory and establish uniform estimates for complex Monge–Ampère equations in families of hermitian varieties.

The complex Monge–Ampère equation is a powerful tool in complex geometry. Many interesting geometric problems (e.g. the Kähler–Einstein equation) can be reduced to such type of equations. Yau's celebrated resolution of the Calabi conjecture [Yau78] and the resolution of Yau–Tian–Donaldson conjecture on the Fano manifolds by Chen–Donaldson–Sun [CDS15] are landmarks in smooth Kähler–Einstein problems.

In recent decades, following the works of Yau [Yau78] and Tsuji [Tsu88], degenerate complex Monge–Ampère equations have been intensively studied. The breakthrough results of Kołodziej [Koł98] and Eyssidieux–Guedj–Zeriahi [EGZ09] led to many important advances. In [EGZ09], Yau's theorem has been generalized to compact Kähler varieties with log terminal singularities. For varieties with ample canonical divisor and semi-log canonical singularities, Berman–Guenancia [BG14] applied the variational approach developed in [BBGZ13] to extend

Aubin–Yau's result [Aub78, Yau78] on stable varieties. On singular Fano varieties, Berman–Boucksom–Jonsson [BBJ21], Li–Tian–Wang [LTW21], and Li [Li22] built a connection between singular Kähler–Einstein metrics and K-stability.

In hermitian contexts, the construction of hermitian Calabi–Yau metrics (i.e. Chern–Ricci flat metrics) is more difficult because the metrics are no longer closed. A Chern–Ricci flat hermitian metric on a complex manifold X can be constructed by solving the complex Monge–Ampère equation

$$(\omega + dd^c\varphi)^n = cf dV_X, \quad \text{and} \quad \sup_X \varphi = 0$$

where

- ω is a smooth $(1,1)$-form,
- dV_X is a smooth volume form on X,
- $f \in L^p(X, dV_X)$ with $p > 1$,

and the pair $(\varphi, c) \in (\mathrm{PSH}(X,\omega) \cap L^\infty(X)) \times \mathbb{R}_{>0}$ is the unknown. When ω is a hermitian metric and f is a smooth positive function, Tosatti–Weinkove [TW10] first showed the existence and uniqueness of the pair (φ, c) with a smooth φ to the above equation. For L^p densities f, Dinew–Kołodziej [DK12] used pluripotential techniques to obtain uniform L^∞-estimates. The solvability was further established by Kołodziej–Nguyen [KN15] via a stability estimate. Recently, Guedj–Lu [GL21] established uniform estimates and proved the existence of solution when the $(1,1)$-form ω is merely big. As a consequence, they generalized Tosatti–Weinkove's theorem to hermitian $\mathbb{Q}$-Calabi–Yau varieties.

It is important to study non-Kähler objects and how special hermitian metrics evolve when complex structures vary. For example, to understand moduli spaces of Calabi–Yau manifolds, a large class of non-Kähler Calabi–Yau threefolds was built via conifold transitions introduced by Clemens and Friedman [Cle83, Fri86]. Reid [Rei87] speculated that all Calabi–Yau threefolds should form a connected family by conifold transitions. Since then, these models attracted a lot of attention (cf. [Fri91, Tia92, Ros06, RZ11, RZ11, Chu12, FLY12, CPY21] and references therein). This is our motivation to study families of singular Chern–Ricci flat metrics.

Uniform L^∞-estimate

Before stating our results, we first fix some geometric setting for families $\pi : \mathcal{X} \to \mathbb{D}$.

Geometric setting (GS). *Let $\mathcal{X}$ be an $(n+1)$-dimensional variety. Suppose that $\pi : \mathcal{X} \to \mathbb{D}$ is a proper surjective holomorphic map with connected fibres $X_t := \pi^{-1}(t)$ which are n-dimensional varieties. Let ω be a hermitian metric on $\mathcal{X}$ in the sense of Definition 4.1.1. For every $t \in \mathbb{D}$, we define a hermitian metric ω_t on the fibre X_t by restriction (i.e. $\omega_t = \omega|_{X_t}$).*

In the sequel, we always assume that families of hermitian varieties $\pi : (\mathcal{X}, \omega) \to \mathbb{D}$ satisfy the geometric setting (GS). We also impose a sup-L^1 comparison (see Conjecture (SL)). Under such conditions, we establish a uniform bound for solutions to complex Monge–Ampère equations in families of hermitian varieties:

Theorem 4.0.1. *Let $\pi : (\mathcal{X}, \omega) \to \mathbb{D}$ be a family of compact, locally irreducible, hermitian varieties and $0 \le f_t \in L^p(X_t, \omega_t^n)$ be a family of densities. Assume that $\pi : (\mathcal{X}, \omega) \to \mathbb{D}$ fits into Conjecture (SL) and $(f_t)_{t\in\mathbb{D}}$ satisfies the following integral bounds: there exist constants $c_f, C_f > 0$ such that for all $t \in \mathbb{D}$,*

$$c_f \le \int_{X_t} f_t^{\frac{1}{n}} \omega_t^n \quad \text{and} \quad \|f_t\|_{L^p(X_t,\omega_t^n)} \le C_f. \tag{IB}$$

For each $t \in \mathbb{D}$, let the pair $(\varphi_t, c_t) \in (\mathrm{PSH}(X_t, \omega_t) \cap L^\infty(X_t)) \times \mathbb{R}_{>0}$ be a solution to the complex Monge–Ampère equation:

$$(\omega_t + dd^c_t \varphi_t)^n = c_t f_t \omega_t^n, \quad \text{and} \quad \sup_{X_t} \varphi_t = 0.$$

Then there exists a constant $C_{\mathrm{MA}} = C_{\mathrm{MA}}(c_f, C_f, C_{SL}, \mathcal{X}, \omega)$ such that for all $t \in \mathbb{D}_{1/2}$,

$$c_t + c_t^{-1} + \|\varphi_t\|_{L^\infty} \le C_{\mathrm{MA}}.$$

Sup-L^1 comparison conjecture

In pluripotential theory, there is a conjecture proposed by Di Nezza–Guedj–Guenancia [DGG20, Conjecture 3.1] which says that if X_0 is irreducible then one has the following sup-L^1 comparison:

Conjecture (SL). *There exists a constant $C_{SL} > 0$ such that: the inequality*

$$\forall \varphi_t \in \mathrm{PSH}(X_t, \omega_t), \quad \sup_{X_t} \varphi_t - C_{SL} \le \frac{1}{V_t} \int_{X_t} \varphi_t \omega_t^n \le \sup_{X_t} \varphi_t$$

holds for all $t \in \mathbb{D}_{1/2}$, where V_t is the volume of X_t with respect to the hermitian metric ω_t.

In the Kähler setting, Di Nezza–Guedj–Guenancia [DGG20, Proposition 3.3] established Conjecture (SL) in the following cases:

(i) The map π is locally trivial or projective;

(ii) The fibres X_t are smooth for $t \neq 0$;

(iii) The fibres X_t have only isolated singularities for every $t \in \mathbb{D}$.

One should notice that the irreducibility of all the fibres is a necessary condition for the left hand side inequality in Conjecture (SL) (cf. [DGG20, Example 3.5]) and it is the reason why we always assume that the fibres are irreducible in the geometric setting (GS).

To establish Conjecture (SL) in hermitian setting, we impose the following assumptions:

Geometric assumption (GA). *Suppose that $\pi : \mathcal{X} \to \mathbb{D}$ is a family of hermitian varieties which satisfies the geometric setting (GS) and one of the following conditions:*

(i) *π is locally trivial;*

(ii) *$\pi : \mathcal{X} \to \mathbb{D}$ is a smoothing of X_0 and X_0 has only isolated singularities.*

Note that both conditions are naturally exclusive unless X_0 is smooth. Also, if $\mathcal{X}$ is smooth and π is a submersion, then (i) holds. Thus, the geometric assumption (GA) includes families of smooth hermitian manifolds. Then we prove that, under the geometric assumption (GA), Conjecture (SL) is fulfilled:

Proposition 4.0.2. *If $\pi : (\mathcal{X}, \omega) \to \mathbb{D}$ is a family of hermitian varieties satisfying the geometric assumption (GA), then there exists a uniform constant C_{SL} such that Conjecture (SL) holds.*

Families of Calabi–Yau varieties

A Calabi–Yau variety X is a normal variety with canonical singularities and trivial canonical bundle K_X. Reid [Rei87] has conjectured that all Calabi–Yau threefolds should form a connected family, provided one allows conifold transitions. Roughly speaking, the construction of a conifold transition goes as follows: contracting a collection of disjoint $(-1,-1)$-curves from a Kähler Calabi–Yau threefold X to get a singular Calabi–Yau variety X_0 and then smoothing singularities of X_0, one obtains a family of Calabi–Yau threefolds $(X_t)_{t\neq 0}$ which are non-Kähler for a general t.

In the model of conifold transitions, the central fibre X_0 has only ordinary double point singularities which are canonical. Based on these geometric models, it is thus legitimate to study a smoothing family of Calabi–Yau varieties where the central fibre has only isolated singularities.

Now, we consider a reasonable "good" family of Calabi–Yau varieties and ask how the bound on the Chern-Ricci potentials vary in families. Assume that $\mathcal{X}$ is a normal variety, $K_{\mathcal{X}}$ is trivial and $\pi : \mathcal{X} \to \mathbb{D}$ is a smoothing. Moreover, we suppose that X_0 has only isolated canonical singularities. One can find a trivializing section Ω of $K_{\mathcal{X}/\mathbb{D}}$. The restriction on each fibre $\Omega_t := \Omega|_{X_t}$ defines a trivialization of K_{X_t}. Following from [GL21, Theorem E], for each t, there is a bounded solution to the corresponding complex Monge–Ampère equation of canonical density. Then we show a uniform estimate in families:

Theorem 4.0.3. *Suppose that $\mathcal{X}$ is normal, $K_{\mathcal{X}}$ is trivial, and $\pi : \mathcal{X} \to \mathbb{D}$ is a smoothing of a variety X_0 whose singularities are canonical and isolated. For each $t \in \mathbb{D}$, let $(\varphi_t, c_t) \in (\mathrm{PSH}(X_t, \omega_t) \cap L^\infty(X_t)) \times \mathbb{R}_{>0}$ be a pair solving the complex Monge–Ampère equation*

$$(\omega_t + dd^c_t \varphi_t)^n = c_t \Omega_t \wedge \overline{\Omega_t} \quad \text{and} \quad \sup_{X_t} \varphi_t = 0.$$

Then there is a uniform constant C_{MA} *such that for all* $t \in \mathbb{D}_{1/2}$

$$c_t + c_t^{-1} + \|\varphi_t\|_{L^\infty} \leq C_{\mathrm{MA}}.$$

Structure of the article

The paper is organized as follows:

- In Section 4.1, we recall basic notions in pluripotential theory and singular spaces.
- Section 4.2 is a recap on methods to obtain L^∞-estimates in local and global cases.
- In Section 4.3, we study the local and global uniform Skoda estimates.
- In Section 4.4, we deal with the volume-capacity comparison stated in Section 4.2.
- In Section 4.5, we establish Conjecture (SL) under the geometric assumption (GA).
- In Section 4.6, we focus on families of Calabi–Yau varieties and show Theorem 4.0.3.

4.1 Preliminaries

In this section, we recall some definitions, notations, and conventions which will be used in the sequel. We define the twisted exterior derivative by $\mathrm{d^c} = \frac{\mathrm{i}}{2}(\bar{\partial} - \partial)$ and we then have $\mathrm{dd^c} = \mathrm{i}\partial\bar{\partial}$. We denote by

- $\mathbb{D}_r := \{z \in \mathbb{C} \,|\, |z| < r\}$ the open disk of radius r;
- $\mathbb{D}_r^* := \{z \in \mathbb{C} \,|\, 0 < |z| < r\}$ the punctured disk of radius r.

When $r = 1$, we simply write $\mathbb{D} := \mathbb{D}_1$ and $\mathbb{D}^* := \mathbb{D}_1^*$.

4.1.1 Smooth forms and currents on singular spaces

Let X be a reduced complex analytic space of pure dimension $n \geq 1$. We denote by X^{reg} the complex manifold of regular points of X and $X^{\mathrm{sing}} := X \setminus X^{\mathrm{reg}}$ the singular set of X. Now, we recall definitions of smooth forms and currents on a complex analytic space X:

Definition 4.1.1. We say that

(i) A smooth form α on X is the data of a smooth form on X^{reg} such that given any local embedding $X \underset{\mathrm{loc.}}{\hookrightarrow} \mathbb{C}^N$, α extends smoothly to $\mathbb{C}^N$;

(ii) A smooth hermitian metric ω on X is a smooth $(1,1)$-form which locally extends to a hermitian metric on $\mathbb{C}^N$;

(iii) $\mathcal{D}_{p,q}(X)$ (resp. $\mathcal{D}_{p,p}(X)_{\mathbb{R}}$) is the space of compactly supported (resp. real) smooth forms of bidegree (p,q);

(iv) The notion of currents, $\mathcal{D}'_{p,q}(X)$ (resp. $\mathcal{D}'_{p,p}(X)_{\mathbb{R}}$), is defined by acting on (resp. real) smooth forms with compact support.

The operators d, $\mathrm{d^c}$ and $\mathrm{dd^c}$ are well-defined by duality (see [Dem85] for details).

4.1.2 Plurisubharmonic functions

Let Ω is an open domain in $\mathbb{C}^n$. We say that u is a *plurisubharmonic* function (*psh* for short) on Ω if it is upper semicontinuous and satisfies the sub-mean inequality on each complex line through every point $x \in \Omega$:

$$u(x) \leq \frac{1}{2\pi}\int_0^{2\pi} u(x+\zeta e^{i\theta})d\theta, \quad \forall x \in \Omega \text{ and } \forall \zeta \in \mathbb{C}^n \text{ such that } |\zeta| < \operatorname{dist}(x, \partial\Omega).$$

We denote by $\mathrm{PSH}(\Omega)$ the space of all psh functions on Ω.

Suppose that X is a reduced complex analytic space equipped with a hermitian metric ω.

Definition 4.1.2. Let $u : X \to [-\infty, \infty)$ be a given function. We say that

(i) u is a psh function on X if it is locally the restriction of a psh function under local embeddings of X into $\mathbb{C}^N$;

(ii) u is *quasi-plurisubharmonic* (*qpsh* for short) on X if it can be locally written as the sum of a psh and a smooth function;

(iii) $\mathrm{PSH}(X, \omega)$ is the set of all *ω-plurisubharmonic* (abbreviated to *ω-psh*), namely, the set of all qpsh functions u which satisfies $\omega + \mathrm{dd^c} u \geq 0$ in the sense of currents.

Remark 4.1.3. There is a weaker notion of (quasi-)plurisubharmonic functions that are defined only as being a locally bounded function on a variety X and its restriction to the complex manifold X^{reg} is (quasi-)plurisubharmonic. On a locally irreducible variety, the stronger notion given above and the weaker notion are equivalent (cf. [Dem85, Théorème 1.7]). In this article, we assume that X also is locally irreducible in some places in order to make sense of the envelope constructions that might not be (quasi-)plurisubharmonic (in the strong sense) on locally reducible complex spaces.

4.1.3 Lelong numbers

Lelong numbers describe the local behavior of currents or psh functions near a point at which it has a log pole. Here we recall a generalized definition given by Demailly:

Definition 4.1.4 ([Dem82, Définition 3])**.** Let X be a complex analytic space. If T is a closed positive (p,p)-current on X and if $x \in X$ is a fixed point, then the Lelong number of T at x is defined as the decreasing limit

$$\nu(T,x) := \lim_{r\to 0} \frac{1}{r^{2(n-p)}} \int_{\{\psi<r\}} T \wedge (\mathrm{dd^c}\psi)^{n-p} = \int_{\{x\}} T \wedge (\mathrm{dd^c} \log \psi)^{n-p}$$

where $\psi = \sum_{i\in I} |g_i|^2$ and $(g_i)_{i\in I}$ is any finite system of generators of the maximum ideal $\mathfrak{m}_{X,x} \subset \mathcal{O}_{X,x}$.

4.1.4 Monge–Ampère capacities

The notion of Monge–Ampère capacities was given by Bedford and Taylor [BT82]. Using Monge–Ampère capacities, they proved that the negligible sets are pluripolar.

Definition 4.1.5. Let $E \subset \Omega$ be a Borel subset. The Bedford–Taylor capacity (or Monge–Ampère capacity) is defined by

$$\mathrm{Cap}(E;\Omega) := \sup\left\{ \int_E (\mathrm{dd^c}u)^n \,\middle|\, u \in \mathrm{PSH}(\Omega) \text{ and } -1 \le u \le 0 \right\}.$$

Theorem 4.1.6 ([BT82])**.** *A subset $E \subset \Omega$ is pluripolar if and only if* $\mathrm{Cap}(E;\Omega) = 0$

For global versions, Kołodziej [Koł03] first defined the Monge–Ampère capacity on a given compact Kähler manifold (X,ω). The definition is analogous to the local cases:

Definition 4.1.7. Let $E \subset X$ be a Borel subset. Define

$$\mathrm{Cap}_\omega(E) := \left\{ \int_E (\omega + \mathrm{dd^c}u)^n \,\middle|\, u \in \mathrm{PSH}(X,\omega) \text{ and } -1 \le u \le 0 \right\}.$$

These definitions can also be extended to non-closed or non-positive form ω and singular complex analytic space X (cf. [Dem85, EGZ09, GZ17, GGZ23]).

4.2 Uniform L^∞-estimate

In this section, we mainly pay attention to L^∞-estimates of complex Monge–Ampère equations on pseudoconvex domains and compact hermitian varieties. We shall follow the method given by Guedj–Kołodziej–Zeriahi [GKZ08] and Guedj–Lu [GL21] to produce a priori estimates. We also compute the precise dependence of these L^∞-estimates on background data.

4.2.1 Local L^∞-estimate

In this section, our goal is to establish a refined version of Kołodziej's a priori estimate [Koł98] of complex Monge–Ampère equation on singular strongly pseudoconvex domain. We recall the definition of a strongly pseudoconvex domain on a Stein space as in [GGZ23, Section 1]. Let S be a singular Stein space which is reduced and locally irreducible, of complex dimension $n \geq 1$. There is a proper embedding $S \hookrightarrow \mathbb{C}^N$ for some N large. A domain $\Omega \Subset S$ is strongly pseudoconvex if it admits a negative smooth psh exhaustion, i.e. a function ρ smooth strongly psh in a neighborhood Ω' of $\overline{\Omega}$ such that $\Omega := \{x \in \Omega' \,|\, \rho(x) < 0\}$. Fix a hermitian metric β on $\mathbb{C}^N$ and define a volume form on S by taking $\mathrm{d}V = \beta^n|_S$.

First, we note that the following estimate always holds:

Volume-capacity comparison (VC). *For every $k > 1$, there exists a constant $C_{VC,k} > 0$ such that*

$$\forall K \Subset \Omega, \quad \mathrm{Vol}(K) \leq C_{VC,k} \,\mathrm{Cap}^k(K;\Omega).$$

The proof of the volume-capacity comparison will be given in Section 4.4 not only in a fixed pseudoconvex set Ω but also for families.

Fix a density $0 \leq f \in L^p(\Omega, \mathrm{d}V)$. Suppose that $\varphi \in \mathrm{PSH}(\Omega) \cap L^\infty(\Omega)$ is the solution to the following Dirichlet problem of complex Monge–Ampère equation

$$\begin{cases} (\mathrm{dd^c}\varphi)^n = f\mathrm{d}V & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases} \tag{locMA}$$

In smooth setting, the existence, uniqueness and the L^∞-estimate of the continuous solution of (locMA) has been constructed by Kołodziej [Koł98]. The existence and uniqueness have been extended by Guedj–Guenancia–Zeriahi [GGZ23, Theorem A] to singular contexts. Now, for singular setup, we prove the following refined version of Kołodziej's L^∞-estimate:

Theorem 4.2.1 (Kołodziej's L^∞-estimate)**.** *Fix $0 \leq f \in L^p(\Omega, \mathrm{d}V)$ with $p > 1$. Suppose that $\varphi \in \mathrm{PSH}(\Omega) \cap L^\infty(\Omega)$ is the solution to the complex Monge–Ampère equation (locMA). Then*

$$\|\varphi\|_{L^\infty} \leq C_{\mathrm{Kol},p} \|f\|_{L^p}^{\frac{1}{n}} \quad \textit{where} \quad C_{\mathrm{Kol},p} := \left[1 + \left(\frac{\mathrm{e}}{1-\mathrm{e}^{-1}}\right) C_{VC,2q}^{1/qn} 2^{1+\frac{1}{q}} \mathrm{Vol}^{\frac{1}{nq^2}}(\Omega) \left(n! C_\rho \|\rho\|_{L^\infty}^n\right)^{\frac{1}{nq}}\right],$$

$1/p + 1/q = 1$, $\mathrm{d}V \leq C_\rho (\mathrm{dd^c}\rho)^n$, and $C_{VC,2q} > 0$ is a constant in the volume-capacity comparison (VC) such that $\mathrm{Vol}(K) \leq C_{VC,2q} \mathrm{Cap}^{2q}(K;\Omega)$ for all $K \Subset \Omega$.

Proof. The idea of proof comes from the work of Guedj–Kołodziej–Zeriahi [GKZ08, Section 1]. We are going to prove the following statement: given $\varepsilon > 0$, we have $\|\varphi\|_{L^\infty(\Omega)} \leq M_\varepsilon$ where

$$M_\varepsilon = \varepsilon + \left(\frac{\mathrm{e}}{1-\mathrm{e}^{-1}}\right) C_{VC,2q}^{1/qn} \left(\frac{2}{\varepsilon}\right)^{1+\frac{1}{q}} \|f\|_{L^p}^{\frac{2}{n}+\frac{1}{nq}} \mathrm{Vol}^{\frac{1}{nq^2}}(\Omega) \left(n! C_\rho \|\rho\|_{L^\infty}^n\right)^{\frac{1}{nq}}.$$

In particular, when $\varepsilon = \|f\|_{L^p}^{\frac{1}{n}}$, one get the desired estimate. Before explaining the proof, we recall some useful facts. For simplicity, we denote by $\mathrm{Cap}(\bullet) = \mathrm{Cap}(\bullet;\Omega)$. First, we recall some basic lemmas:

Lemma 4.2.2. *Fix $\varphi, \psi \in \mathrm{PSH}(\Omega) \cap L^\infty(\Omega)$ such that $\liminf_{z \to \partial\Omega}(\varphi - \psi) \geq 0$. Then for all $t, s > 0$ we have*

$$t^n \operatorname{Cap}(\{\varphi - \psi < -s - t\}) \leq \int_{\{\varphi - \psi < -s\}} (\mathrm{dd^c}\varphi)^n.$$

By definition, for all $u \in \mathrm{PSH}(\Omega) \cap L^\infty(\Omega)$, complex Monge–Ampère measures $(\mathrm{dd^c}u)^n$ put zero mass on Ω^{sing}; hence we still have the comparison principle on Ω (cf. [GGZ23, Proposition 1.5]). Then one can follow exactly the same argument in [GKZ08, Lemma 1.3] to obtain Lemma 4.2.2.

Using the volume-capacity comparison (VC) and Hölder's inequality, one has the estimate as follows

Lemma 4.2.3. *For all $\tau > 1$, there exists a constant $D_\tau := C_{VC,k}^{\frac{p-1}{p}} \|f\|_{L^p}$ where $k = k(\tau, p) := \frac{\tau p}{(p-1)} = \tau q$ such that*

$$\forall K \Subset \Omega, \quad 0 \leq \int_K f \mathrm{d}V \leq D_\tau \operatorname{Cap}^\tau(K).$$

The following classical lemma is due to De Giorgi and the reader is referred to [GKZ08, Lemma 1.5] and [EGZ09, Lemma 2.4] for the proof.

Lemma 4.2.4. *Let $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a decreasing right-continuous function and satisfy $\lim_{s\to\infty} g(s) = 0$. Assume that there exists $\tau > 1, B > 0$ such that g satisfies*

$$\forall s, t > 0, \quad tg(s+t) \leq B(g(s))^\tau.$$

Then $g(s) = 0$ for all $s \geq s_\infty$, where

$$s_\infty = \frac{\mathrm{e}B(g(0))^{\tau-1}}{1 - \mathrm{e}^{1-\tau}}.$$

By Lemma 4.2.2 and Lemma 4.2.3, we have

$$t^n \operatorname{Cap}(\{\varphi < -s - t\}) \leq \int_{\{\varphi < -s\}} (\mathrm{dd^c}\varphi)^n = \int_{\{\varphi < -s\}} f \mathrm{d}V \leq D_2 \operatorname{Cap}^2(\{\varphi < -s\})$$

where $D_2 = C_{VC,2q}^{1/q} \|f\|_{L^p}$. Thus,

$$t \operatorname{Cap}^{1/n}(\{\varphi < -s - t\}) \leq D_2^{1/n} \operatorname{Cap}^{2/n}(\{\varphi < -s\}).$$

Let $g(s) := \operatorname{Cap}^{1/n}(\{\varphi < -s - \varepsilon\})$. Then we have $tg(s+t) \leq Bg(s)^2$ where $B = D_2^{1/n}$. Using Lemma 4.2.4, one obtains $\operatorname{Cap}(\{\varphi < -s - \varepsilon\}) = 0$ for all $s \geq s_\infty = \frac{\mathrm{e}Bg(0)}{1-\mathrm{e}^{-1}}$. This implies that $\varphi \geq -s_\infty - \varepsilon$ almost everywhere and thus everywhere by plurisubharmonicity. Therefore, one has

$$\sup_\Omega(-\varphi) \leq \varepsilon + s_\infty = \varepsilon + \frac{\mathrm{e}Bg(0)}{1 - \mathrm{e}^{-1}}.$$

Now, we need to control $g(0) = \mathrm{Cap}^{1/n}(\{\varphi < -\varepsilon\})$. By Lemma 4.2.2 and Chebyshev inequality for a fixed constant $r > 0$, we have

$$\left(\frac{\varepsilon}{2}\right)^n \mathrm{Cap}\left(\left\{\varphi < -\frac{\varepsilon}{2} - \frac{\varepsilon}{2}\right\}\right) \leq \int_{\{\varphi<-\varepsilon/2\}} (\mathrm{dd^c}\varphi)^n = \int_{\{\varphi<-\varepsilon/2\}} f\mathrm{d}V$$
$$\leq \int_\Omega \left(\frac{-2\varphi}{\varepsilon}\right)^r f\mathrm{d}V \leq \|f\|_{L^p} \left(\int_\Omega \left(\frac{-2\varphi}{\varepsilon}\right)^{rq} \mathrm{d}V\right)^{1/q}.$$

Put $r = n/q$. Note that integration by parts is legitimate in our setting (see [DGG20, Lemma 2.11]). By Błocki's estimate of integration by parts [Bło93, Theorem 2.1] and Hölder inequality, one can derive that

$$\int_\Omega (-\varphi)^n \mathrm{d}V \leq C_\rho \int_\Omega (-\varphi)^n (\mathrm{dd^c}\rho)^n$$
$$\leq n! C_\rho \|\rho\|^n_{L^\infty} \int_\Omega (\mathrm{dd^c}\varphi)^n \leq n! C_\rho \|\rho\|^n_{L^\infty} \mathrm{Vol}^{\frac{1}{q}}(\Omega) \|f\|_{L^p}.$$

One can infer

$$g(0)^n = \mathrm{Cap}(\{\varphi < -\varepsilon\}) \leq \left(\frac{2}{\varepsilon}\right)^{n+\frac{n}{q}} \|f\|^{1+\frac{1}{q}}_{L^p} \mathrm{Vol}^{\frac{1}{q^2}}(\Omega) \left(n! C_\rho \|\rho\|^n_{L^\infty}\right)^{1/q}.$$

All in all, we obtain the desired estimate:

$$\|\varphi\|_{L^\infty} \leq \varepsilon + \left(\frac{\mathrm{e}}{1-\mathrm{e}^{-1}}\right) C^{1/qn}_{VC,2q} \left(\frac{2}{\varepsilon}\right)^{1+\frac{1}{q}} \|f\|^{\frac{2}{n}+\frac{1}{nq}}_{L^p} \mathrm{Vol}^{\frac{1}{nq^2}}(\Omega) \left(n! C_\rho \|\rho\|^n_{L^\infty}\right)^{\frac{1}{nq}}.$$

□

4.2.2 Global L^∞-estimate

Suppose that (X, ω) is an n-dimensional compact locally irreducible hermitian variety. Fix a function $0 \leq f \in L^p(X, \omega^n)$ with $p > 1$. First, we fix some notations:

(i) Denote the volume of X with respect to ω^n by V;

(ii) A constant $B' > 0$ is such that $-B'\omega^n \leq \mathrm{dd^c}\omega^{n-1} \leq B'\omega^n$;

(iii) Fix a finite double cover $(\Omega_j := \{\rho_j < 0\})_{1\leq j\leq N}$ and $(\Omega'_j := \{\rho_j < -c_j\})_{1\leq j\leq N}$ of X where for each j, the function ρ_j is smooth on X, strictly psh near $\overline{\Omega}_j$, and $0 \leq \rho_j \leq 1$ on $X \setminus \Omega_j$, and $c_j > 0$ is a constant;

(iv) $C_{\mathrm{Kol},p}$ is a constant such that $\|u_j\|_{L^\infty(\Omega_j)} \leq C_{\mathrm{Kol},p} \|f\|^{\frac{1}{n}}_{L^p(\Omega_j,\omega^n)}$ where for each $1 \leq j \leq N$, the function u_j is the solution to the Dirichlet problem

$$\begin{cases} (\mathrm{dd^c} u_j)^n = f\omega^n & \text{in } \Omega_j, \\ u_j = 0 & \text{on } \partial\Omega_j. \end{cases}$$

(v) A_ρ is a constant such that $A_\rho\omega + \mathrm{dd^c}\rho_j > 0$ on X for all $1 \leq j \leq N$;

(vi) $c_\rho = \min_{1\leq j\leq N} c_j > 0$.

In this section, we fix some geometric constants and impose two integral bounds on the density f:

Geometric constants (SL). *There is a constant $C_{SL} > 0$ such that the following inequality holds*

$$\forall \varphi \in \mathrm{PSH}(X,\omega), \quad \sup_X \varphi - C_{SL} \leq \frac{1}{V}\int_X \varphi\omega^n \leq \sup_X \varphi;$$

Geometric constants (Skoda). *There exist $\alpha > 0$ and $A_\alpha > 0$ such that*

$$\forall u \in \mathrm{PSH}(X,\omega), \quad \int_X \mathrm{e}^{\alpha(\sup_X u - u)}\omega^n \leq A_\alpha.$$

Analytic constants (AC). *Let $0 \leq f \in L^p(X,\omega)$ for some $p > 1$. There are two constants $c_f, C_f > 0$ such that*

$$c_f \leq \int_X f^{\frac{1}{n}}\omega^n \quad \text{and} \quad \|f\|_{L^p} \leq C_f.$$

Following the strategy in [GL21], we shall prove an a priori L^∞-estimate of complex Monge–Ampère equations on hermitian varieties.

Theorem 4.2.5. *Let $(\varphi, c) \in (\mathrm{PSH}(X,\omega) \cap L^\infty(X)) \times \mathbb{R}_{>0}$ be a pair solving the complex Monge–Ampère equation*

$$(\omega + \mathrm{dd^c}\varphi)^n = cf\omega^n \quad \text{and} \quad \sup_X \varphi = 0. \tag{MA}$$

There exists a uniform positive constant $C_{\mathrm{MA}} > 0$ such that

$$c + c^{-1} + \|\varphi\|_{L^\infty(X)} \leq C_{\mathrm{MA}}$$

and the constant C_{MA} depends only on n, V, B', N, A_ρ, c_ρ, α, A_α, C_{Kol}, p, c_f, C_f, and C_{SL}.

Remark 4.2.6. Suppose that X is a compact normal variety and $\mu : \widetilde{X} \to X$ is a resolution of singularities. Following [GL21, Theorem B], there exists a pair (φ, c) solving the corresponding complex Monge–Ampère equation on $\widetilde{X}$. Let $E = \mathrm{Exc}(\mu)$ be the exceptional divisor of μ which is analytic and thus pluripolar. Since the complex Monge–Ampère measure $(\mu^*\omega + \mathrm{dd^c}\varphi)^n$ charges no mass on E, one can descend the solution to X^{reg}. By the extension theorem of Grauert and Remmert [GR56, Satz 4] and normality of X, the $\mu_*\varphi$ induces a ω-psh function on X and it solves the complex Monge–Ampère equation.

Upper bound of c

Following the same idea as in [KN15, Lemma 5.9], one can find an upper bound of c simply using the arithmetic-geometric mean inequality:

Lemma 4.2.7. *With the geometric constant C_{SL} in (SL), one has*

$$c \leq \left(\frac{C_{\mathrm{Lap}}}{nc_f}\right)^n \quad \text{where } C_{\mathrm{Lap}} := V(1 + B'C_{SL}).$$

Proof. For all $u \in \mathrm{PSH}(X, \omega)$, we compute

$$\int_X (\omega + dd^c u) \wedge \omega^{n-1} = \int_X \omega^n + \int_X (u - \sup_X u) dd^c \omega^{n-1} \leq V(1 + B'C_{SL}) =: C_{\mathrm{Lap}}.$$

Then applying arithmetic-geometric mean inequality, one can infer that

$$c^{1/n} \int_X f^{1/n} \omega^n = \int_X \left(\frac{(\omega + dd^c\varphi)^n}{\omega^n}\right)^{\frac{1}{n}} \omega^n \leq \frac{1}{n} \int_X (\omega + dd^c \varphi) \wedge \omega^{n-1} \leq \frac{1}{n} C_{\mathrm{Lap}}.$$

Rearranging the inequality, we obtain an upper bound of c as desired. □

Domination principle

The domination principle has been proved under several setup (cf. [Ngu16, LPT21, GL22, GL21] and references therein). Here we establish the following domination principle on singular varieties:

Lemma 4.2.8 (Domination principle). *Let $u, v \in \mathrm{PSH}(X, \omega) \cap L^\infty(X)$. Then we have the following properties*

(i) if $(\omega + dd^c u)^n \leq c(\omega + dd^c v)^n$, then $c \geq 1$;

(ii) if $e^{-\lambda u}(\omega + dd^c u)^n \leq e^{-\lambda v}(\omega + dd^c v)^n$ for some $\lambda > 0$, then $v \leq u$.

Sketch of proof. Let $\mu : \widetilde{X} \to X$ be a resolution of singularities. The $(1,1)$-form $\mu^*\omega$ is semi-positive and big (cf. [GL21, Definition 1.6]). For all functions $u \in \mathrm{PSH}(X, \omega) \cap L^\infty(X)$, the Monge–Ampère measure $(\mu^*\omega + dd^c \mu^* u)^n$ puts no mass on the exceptional divisor $E := \mathrm{Exc}(\mu)$ since E is a pluripolar set. On the other hand, by definition, $(\omega + dd^c u)^n$ charges no mass on X^{sing} for all $u \in \mathrm{PSH}(X, \omega) \cap L^\infty(X)$. Hence, one can descend the domination principles in [GL21, Corollary 1.13 and 1.14] to X. Then one can conclude Lemma 4.2.8. □

Subsolution estimate

In [GL21, Theorem 2.1], Guedj and Lu constructed a subsolution (ψ, m) to the complex Monge–Ampère equation with a given L^p-density g. We follow the same method to get subsolution estimate on singular varieties while also carefully keeping track of the dependence of data.

Proposition 4.2.9. *For all $p > 1$, there exist uniform constants $m_p, M_p > 0$ such that for every $0 \le g \in L^p(X, \omega)$ with $\|g\|_{L^p} = 1$, there is a function $\psi \in \mathrm{PSH}(X, \omega) \cap L^\infty(X)$ satisfying*

$$(\omega + dd^c\psi)^n \ge m_p g\omega^n \quad \textit{and} \quad \mathrm{osc}_X\, \psi \le M_p.$$

Precisely, the constants m_p and M_p can be taken as

$$m_p = \frac{c_\rho}{NA_\rho\left(C_{\mathrm{Kol},p} + 1\right)} \quad \textit{and} \quad M_p = C_{\mathrm{Kol},p}\left(\frac{1}{A_\rho} + 1\right).$$

Proof. For each j, let the function $u_j \in \mathrm{PSH}(\Omega_j) \cap L^\infty(\Omega_j)$ be the unique solution to the following complex Monge–Ampère equation

$$\begin{cases} (dd^c u_j)^n = g\omega^n & \text{in } \Omega_j, \\ u_j = -1 & \text{on } \partial\Omega_j. \end{cases}$$

From Theorem 4.2.1, there is a constant $C_{\mathrm{Kol},p} > 0$ such that for all j,

$$\|u_j\|_{L^\infty} \le C_{\mathrm{Kol},p}\, \|g\|^{1/n}_{L^p(\Omega_j, \omega^n)} + 1 \le C_{\mathrm{Kol},p} + 1.$$

Now, we consider the psh functions $(v_j)_j$ defined by $v_j := \max\left\{u_j, \frac{C_{\mathrm{Kol},p}+1}{c_j}\rho_j\right\}$ for each j. One can see that the following properties are satisfied

- $v_j = u_j$ in Ω'_j and $(dd^c v_j)^n = g\omega^n$ in Ω'_j;

- $v_j = \frac{C_{\mathrm{Kol},p}+1}{c_j}\rho_j$ on $X \setminus \Omega_j$ and near the boundary of $\partial\Omega_j$.

From the construction, one can check that for every j, $\left(\frac{A_\rho(C_{\mathrm{Kol},p}+1)}{c_\rho}\right)\omega + dd^c v_j \ge 0$ as well. We define the subsolution ψ as follows

$$\psi = \frac{c_\rho}{NA_\rho\left(C_{\mathrm{Kol},p} + 1\right)} \sum_{j=1}^{N} v_j.$$

Note that in Ω'_j, we have

$$(\omega + \mathrm{dd^c}\psi)^n = \left(\frac{c_\rho}{NA_\rho\left(C_{\mathrm{Kol},p}+1\right)} \sum_{j=1}^{N} \left[\frac{A_\rho\left(C_{\mathrm{Kol},p}+1\right)}{c_\rho}\omega + \mathrm{dd^c} v_j \right] \right)^n$$
$$\geq \frac{c_\rho}{NA_\rho\left(C_{\mathrm{Kol},p}+1\right)} (\mathrm{dd^c} u_j)^n = \frac{c_\rho}{NA_\rho\left(C_{\mathrm{Kol},p}+1\right)} g\omega^n.$$

Then we derive that

$$\psi \leq \frac{1}{A_\rho} \quad \text{and} \quad \psi \geq -C_{\mathrm{Kol},p} - 1 \implies \mathrm{osc}_X\,\psi \leq \frac{1}{A_\rho} + C_{\mathrm{Kol},p} + 1.$$

□

L^∞-estimate

We now prove an a priori L^∞-estimate following the method in [GL21, Theorem 2.1].

Theorem 4.2.10. *Let (X,ω) be a compact hermitian variety with $\dim_{\mathbb{C}} X = n$. Fix a density $0 \leq f \in L^p(X, \omega^n)$. Assume that $c_f \leq \int_X f^{\frac{1}{n}}\omega^n$ for a constant $c_f > 0$. Let the pair $(\varphi, c) \in (\mathrm{PSH}(X,\omega) \cap L^\infty(X)) \times \mathbb{R}_{>0}$ be a solution to (MA). Then one has*

$$c \geq \frac{m_{\frac{p+1}{2}}}{A_\alpha^{\frac{p-1}{p(p+1)}} C_f}, \quad c \leq \left(\frac{C_{\mathrm{Lap}}}{nc_f}\right)^n, \quad \text{and} \quad \|\varphi\|_{L^\infty} \leq M_{\frac{p+1}{2}} + \frac{2p(p+1)}{\alpha(p-1)} \left(\log \frac{A_\alpha^{\frac{p-1}{p(p+1)}} C_f C_{\mathrm{Lap}}^n}{m(nc_f)^n} \right).$$

Proof. We consider a twisted function $g' = \mathrm{e}^{-\varepsilon\varphi} f$ for some $\varepsilon > 0$. Fix constants $p' = \frac{p+1}{2} \in (1,p)$ and $\varepsilon = \frac{\alpha(p-1)}{2p(p+1)} = \frac{\alpha(p-p')}{2pp'} \in \left(0, \frac{\alpha(p-p')}{pp'}\right)$. One can derive that $g' \in L^{p'}$. Indeed, by Hölder inequality, we have

$$\|g'\|_{L^{p'}} \leq \left\|\mathrm{e}^{-\varepsilon\varphi}\right\|_{L^{\frac{pp'}{p-p'}}} \|f\|_{L^p} \leq A_\alpha^{\frac{p-p'}{pp'}} \|f\|_{L^p}.$$

Put $g = g'/\|g'\|_{L^{p'}}$. From Proposition 4.2.9, we have a bounded ω-psh function ψ with $\sup_X \psi = 0$ such that

$$(\omega + \mathrm{dd^c}\psi)^n \geq m_{p'} g\omega^n = m_{p'} \frac{\mathrm{e}^{-\varepsilon\varphi} f}{\|g'\|_{L^{p'}}}\omega^n = \frac{m_{p'}\mathrm{e}^{-\varepsilon\varphi}}{c\,\|g'\|_{L^{p'}}} cf\omega^n = \frac{m_{p'}\mathrm{e}^{-\varepsilon\varphi}}{c\,\|g'\|_{L^{p'}}}(\omega + \mathrm{dd^c}\varphi)^n$$
$$\geq \frac{m_{p'}}{c\,\|g'\|_{L^{p'}}}(\omega + \mathrm{dd^c}\varphi)^n.$$

and $\|\psi\|_{L^\infty} \leq M_{\frac{p+1}{2}}$. By Lemma 4.2.8, one get $\frac{m_{p'}}{c\|g'\|_{L^{p'}}} \leq 1$; hence c has a lower bound, $c \geq$

$\frac{m_{p'}}{A_\alpha^{\frac{p-p'}{pp'}}\|f\|_{L^p}}$. Also, we see that

$$e^{-\varepsilon\psi}(\omega+dd^c\psi)\geq\frac{m_{p'}e^{-\varepsilon\varphi}}{c\|g'\|_{L^{p'}}}(\omega+dd^c\varphi)^n=\exp\left(-\varepsilon\left(\varphi-\frac{1}{\varepsilon}\log\frac{m_{p'}}{c\|g'\|_{L^{p'}}}\right)\right)(\omega+dd^c\varphi)^n.$$

Applying the domination principle again, one can infer

$$M_{p'}\leq\psi\leq\varphi-\frac{1}{\varepsilon}\left(\log\frac{m_{p'}}{c\|g'\|_{L^{p'}}}\right)$$

Finally, this provides a uniform L^∞-estimate as follows

$$\|\varphi\|_{L^\infty}\leq M_{p'}-\frac{1}{\varepsilon}\left(\log\frac{m}{c\|g'\|_{L^{p'}}}\right)\leq M_{p'}+\frac{1}{\varepsilon}\left(\log\frac{A_\alpha^{\frac{p-p'}{pp'}}\|f\|_{L^p}C_{\mathrm{Lap}}^n}{m(nc_f)^n}\right).$$

□

4.2.3 Proof of Theorem 4.0.1

Recall that the constant C_{MA} depends only on n, V, B', N, A_ρ, c_ρ, α, A_α, C_{Kol}, p, c_f, C_f, and C_{SL}. We shall control these constant to get Theorem 4.0.1. The following data are included in the assumption of Theorem 4.0.1:

- n is fixed in the geometric setting (GS);
- p, c_f, C_f are data in the integral bounds (IB);
- C_{SL} is given by Conjecture (SL).

Then fixing some choices of background data, we have uniform control of the following constants:

- A_ρ, c_ρ, N: After shrinking $\mathbb{D}$, we can cover $\mathcal{X}$ by finitely many pseudoconvex double cover $(\mathcal{U}_j=\{\rho_j<0\})_j$ and $(\mathcal{U}_j'=\{\rho_j<-c_j\})_j$. Then the slices $\Omega_j:=\mathcal{U}_j\cap X_t=\{\rho_j|_{X_t}<0\}$ and $\Omega_j':=\mathcal{U}_j'\cap X_t=\{\rho_j|_{X_t}<-c_j\}$ form a pseudoconvex double cover of X_t for all $t\in\mathbb{D}_{1/2}$. Hence, these constants are fixed under such choice of a double covering;
- B': It can be obtained easily by restriction on each fibres;
- V: It follows from continuity of the total mass of the currents $(\omega^n\wedge[X_t])_{t\in\overline{\mathbb{D}}_{1/2}}$ (cf. [Pan22, Section 1.4]).

The remaining data is the main focus of Sections 4.3 and 4.4:

- α, A_α: These would be established in Proposition 4.3.3 assuming Conjecture (SL);

- $C_{\mathrm{Kol},p}$: A uniform version of the volume-capacity comparison (VC) will be treated by Proposition 4.4.3 and thus using Theorem 4.2.1 and uniform control of $\rho_j|_{X_t}$ for all $t \in \mathbb{D}_{1/2}$, one can obtain a uniform constant $C_{\mathrm{Kol},p}$.

These complete the proof of Theorem 4.0.1.

4.3 Uniform Skoda's integrability theorem

In this section, we follow the ideas in [Zer01, DGG20] to establish a local version of uniform Skoda's integrability theorem in families. Then we prove a uniform global version of Skoda's integrability theorem (i.e. geometric constants in (Skoda)) in a family which has a uniform $C_{SL} > 0$.

4.3.1 Local uniform Skoda's estimate

Recall that $\pi : \mathcal{X} \to \mathbb{D}$ is a proper surjective holomorphic map and satisfies the geometric setting (GS). Then we fix some notations:

(i) $\mathcal{U}$ is a strongly pseudoconvex domain in $\mathcal{X}$ and $\mathcal{U}$ which is contained in a larger strongly pseudoconvex domain $\widetilde{\mathcal{U}} \subset \mathcal{X}$ such that $\pi(\widetilde{\mathcal{U}}) \Subset \mathbb{D}$;

(ii) ρ (resp. $\widetilde{\rho}$) is a smooth strictly psh function defined in a neighborhood of $\overline{\mathcal{U}}$ (resp. $\overline{\widetilde{\mathcal{U}}}$) such that $\mathcal{U} := \{\rho < 0\}$ (resp. $\widetilde{\mathcal{U}} := \{\widetilde{\rho} < 0\}$) and $\mathcal{U}_c = \{\rho < -c\}$ is relatively compact in $\mathcal{U}$ for each $c > 0$;

(iii) Fix a relatively compact subdomain $\mathcal{U}' \Subset \mathcal{U}$ such that $\mathcal{U}' := \{\rho < -c\}$ for some generic $c > 0$ with $\mathrm{d}\rho$ non-vanishing on $\partial\mathcal{U}'$;

(iv) Define the slices by $\Omega_t := \mathcal{U} \cap X_t$, $\Omega'_t := \mathcal{U}' \cap X_t$, and $\widetilde{\Omega}_t := \widetilde{\mathcal{U}} \cap X_t$, which are strongly pseudoconvex domains in X_t.

(v) Fix ω a hermitian metric which comes from a restriction of a hermitian metric in $\mathbb{C}^N$. Let $\mathrm{d}V_t = \omega^n|_{X_t}$ be a smooth volume form defined on $\widetilde{\Omega}_t$.

Under such setup, we show the following Skoda-type estimate, independently of t.

Theorem 4.3.1. *Let $\mathcal{F}_t$ be a family of negative psh functions defined on $\widetilde{\Omega}_t$. Fix $r > 0$ such that $\mathbb{D}_r \Subset \pi(\mathcal{U}) \Subset \mathbb{D}$. Assume that there is a uniform constant $C_F > 0$ such that for all $u_t \in \mathcal{F}_t$,*

$$\int_{\Omega_t} (-u_t)\mathrm{d}V_t \le C_F.$$

Then there exist positive constants α and A_α which depend on C_F such that for each $u_t \in \mathcal{F}_t$,

$$\int_{\Omega'_t} \mathrm{e}^{-\alpha u_t}\mathrm{d}V_t \le A_\alpha.$$

for all $t \in \mathbb{D}_r$.

Remark 4.3.2. We mostly use $\mathcal{U}$ and $\mathcal{U}'$ in the proof. However, in order to approximate a negative psh function by a decreasing sequence of smooth psh functions, we need to shrink the domain in the very beginning of the proof. That is the reason why we choose $\mathcal{U}$ lying in a bigger pseudoconvex domain $\widetilde{\mathcal{U}}$.

Proof. We proceed in several steps:

Step 0: Good covers. First of all, we may assume that u_t are smooth. It follows indeed from a result of Fornaess–Narasimhan [FN80, Theorem 5.5], that one can approximate u_t by a decreasing sequence of smooth non-positive psh functions on $\mathcal{U} \cap X_t$. We set

$$v_t := u_t + \rho_t$$

where $\rho_t = \rho|_{\Omega_t}$. Since the masses of $\beta^n \wedge [X_t]$ are continuous in t, $\mathrm{Vol}(\Omega_t, \mathrm{d}V_t)$ is uniformly bounded by a constant $V' > 0$ up to shrinking $\mathbb{D}$ (cf. [Pan22, Section 1.4]). By adding $\|\rho\|_{L^\infty(\mathcal{U})} V'$ to C_F, we can also assume that

$$\forall u_t \in \mathcal{F}_t, \quad \int_{\Omega_t} (-v_t)\mathrm{d}V_t \le C_F.$$

Choose a finite collection of balls of radius 2, $B_{\mathbb{C}^N}(p, 2) \subset \mathbb{C}^N$, centered at some point $p \in X_0$. We denote by $\mathbb{B}_R := \mathcal{X} \cap B_{\mathbb{C}^N}(p, R) \Subset \mathcal{U}$ for all $R \le 2$. One may assume that the collection of balls of radius 1/2, $\{\mathbb{B}_{1/2}\}$, covers Ω_t' for all $t \in \overline{\mathbb{D}}_r$. For convenience, in this section, we fix constants $C_\rho, C_\omega > 0$ satisfying $C_\rho^{-1}\mathrm{dd^c}\rho \le \mathrm{dd^c}|z|^2 \le C_\rho \mathrm{dd^c}\rho$ and $C_\omega^{-1}\omega \le \mathrm{dd^c}|z|^2 \le C_\omega \omega$ on each $\mathbb{B}_2$

Step 1: Poisson–Szegő inequality. We first recall the following inequality

$$\begin{aligned} v_t(x) &\ge \int_{\mathbb{B}} v_t(\mathrm{dd^c}G_x)^n \wedge [X_t] = \int_{\mathbb{B}\cap X_t} v_t(\mathrm{dd^c}G_x)^n \\ &= \underbrace{\int_{\mathbb{B}\cap X_t} G_x(\mathrm{dd^c}v_t) \wedge (\mathrm{dd^c}G_x)^{n-1}}_{:=I_t(x)} + \underbrace{\int_{\partial\mathbb{B}\cap X_t} v_t \mathrm{d^c}G_x \wedge (\mathrm{dd^c}G_x)^{n-1}}_{:=J_t(x)}, \end{aligned}$$

where $G_x(z) := \log|\Phi_x(z)|$ and $\Phi_x(z)$ is the automorphism of the unit ball $\mathbb{B}$ that sends x to the origin. The reader is referred to [DGG20, page 22-23] for more details.

Step 2: Control J_t, I_t and Lelong numbers. Following the same proof in [DGG20, middle of page 23], we have $|J_t| \le C_F C_1$ for some uniform $C_1 > 0$. Now, we are going to treat the other more singular term

$$I_t(x) = \int_{\mathbb{B}} G_x(\mathrm{dd^c}v_t) \wedge (\mathrm{dd^c}G_x)^{n-1} \wedge [X_t].$$

In global Kähler setting, this part can be controlled by cohomology class of given Kähler metrics but it is not the case here. The spirit goes back to the local strategy in [Zer01] and Chern–

Levine–Nirenberg inequality.

Consider the mass of the measure $dd^c v_t \wedge (dd^c G_x)^{n-1} \wedge [X_t]$

$$\begin{aligned}\gamma_t(x) &:= \int_{\mathbb{B}} dd^c v_t \wedge (dd^c G_x)^{n-1} \wedge [X_t] \\ &= \underbrace{\int_{D(x,r)} dd^c v_t \wedge (dd^c G_x)^{n-1} \wedge [X_t]}_{:=I_t'(x)} + \underbrace{\int_{\mathbb{B}\setminus D(x,r)} dd^c v_t \wedge (dd^c G_x)^{n-1} \wedge [X_t]}_{:=I_t''(x)}\end{aligned}$$

where $D(x,r) := \{\zeta \in \mathbb{B} \,|\, |\Phi_x(\zeta)| < r\}$ for some $r \in (0,1)$. Note that $\mu_t = \frac{1}{\gamma_t} dd^c v_t \wedge (dd^c G_x)^{n-1} \wedge [X_t]$ is a probability measure.

From the definition of $G_x(z)$, direct computation provides that

$$dd^c G_x(z) \le C_2 \frac{dd^c|z|^2}{|\Phi_x(z)|^2} \tag{4.3.1}$$

for some uniform constant $C_2 > 0$. Choose a cut-off function χ supported in $\mathbb{B}_2$ and satisfying $\chi \equiv 1$ on $\mathbb{B}$ and $-C_3 dd^c|z|^2 \le dd^c\chi \le C_3 dd^c|z|^2$ for some constant $C_3 > 0$. Then we have

$$\begin{aligned}I_t''(x) &\le \frac{1}{r^{2n-2}} \int_{\mathbb{B}} dd^c v_t \wedge (dd^c|z|^2)^{n-1} \wedge [X_t] \\ &\le \frac{1}{r^{2n-2}} \int_{\mathbb{B}_2} \chi dd^c v_t \wedge (dd^c|z|^2)^{n-1} \wedge [X_t] = \frac{1}{r^{2n-2}} \int_{\mathbb{B}_2} v_t dd^c\chi \wedge (dd^c|z|^2)^{n-1} \wedge [X_t] \\ &\le \frac{C_3}{r^{2n-2}} \int_{\mathcal{U}} (-v_t)(dd^c|z|^2)^n \wedge [X_t] = \frac{C_3}{r^{2n-2}} \int_{\Omega_t} (-v_t)(dd^c|z|^2)^n \le \frac{C_3 C_F}{r^{2n-2}}.\end{aligned}$$

Because $|x| < 1/2$ in the setting, one may assume $D(x, r_0) \subset \mathbb{B}_{3/4}$ for some uniform $r_0 > 0$ sufficiently small. Hence, one get $I_t''(x) \le \frac{C_3 C_F}{r_0^{2n-2}}$.

Consider cut-offs $(\chi_j)_{j=1}^n$ which are compactly supported on $\mathbb{B}$ and satisfy: $\chi_1 \equiv 1$ on $\mathbb{B}_{3/4}$, $\mathrm{supp}(\chi_1) \Subset \mathbb{B}$, $\chi_{j+1} \equiv 1$ on $\mathrm{supp}(\chi_i)$ for every $j \in \{1, ..., n-1\}$ and $-C_4 dd^c|z|^2 \le dd^c\chi_j \le C_4 dd^c|z|^2$ for some constant $C_4 > 0$. Using the trick in Chern–Levine–Nirenberg inequality, one can see that

$$
\begin{aligned}
I_t'(x) &\le \int_{\mathbb{B}_{3/4}} dd^c v_t \wedge (dd^c G_x)^{n-1} \wedge [X_t] \le \int_{\mathrm{supp}(\chi_1)} \chi_1 dd^c v_t \wedge (dd^c G_x)^{n-1} \wedge [X_t] \\
&\le \int_{\mathrm{supp}(\chi_1)} G_x dd^c v_t \wedge dd^c \chi_1 \wedge (dd^c G_x)^{n-2} \wedge [X_t] \\
&\le C_4 \left(\sup_{z \in \mathbb{B} \setminus \mathbb{B}_{3/4}} |G_x(z)| \right) \int_{\mathrm{supp}(\chi_1)} dd^c v_t \wedge dd^c |z|^2 \wedge (dd^c G_x)^{n-2} \wedge [X_t] \\
&\le C_4 \left(\sup_{z \in \mathbb{B} \setminus \mathbb{B}_{3/4}} |G_x(z)| \right) \int_{\mathrm{supp}(\chi_2)} \chi_2 dd^c v_t \wedge dd^c |z|^2 \wedge (dd^c G_x)^{n-2} \wedge [X_t] \\
&\le \cdots \\
&\le C_4^{n-1} \left(\sup_{z \in \mathbb{B} \setminus \mathbb{B}_{3/4}} |G_x(z)| \right)^{n-1} \int_{\mathrm{supp}(\chi_{n-1})} dd^c v_t \wedge (dd^c |z|^2)^{n-1} \wedge [X_t] \\
&\le C_4^{n} \left(\sup_{z \in \mathbb{B} \setminus \mathbb{B}_{3/4}} |G_x(z)| \right)^{n-1} \int_{\mathbb{B}} (-v_t) \wedge (dd^c |z|^2)^{n} \wedge [X_t] \\
&\le C_4^{n} \left(\sup_{z \in \mathbb{B} \setminus \mathbb{B}_{3/4}} |G_x(z)| \right)^{n-1} \int_{\Omega_t} (-v_t) \wedge (dd^c |z|^2)^{n} \le C_4^n C_F \left(\sup_{z \in \mathbb{B} \setminus \mathbb{B}_{3/4}} |G_x(z)| \right)^{n-1}.
\end{aligned}
$$

Since $|x| < 1/2$ and $\Phi_x(z)$ moves smoothly in x and z, there is a uniform constant C_5 such that

$$
\left(\sup_{z \in \mathbb{B} \setminus \mathbb{B}_{3/4}} |G_x(z)| \right) \le C_5.
$$

Thus, I_t' is uniformly bounded from above by the constant $C_4^n C_F C_5^{n-1}$.

Estimates of I' and I'' yield a constant $\nu = \frac{C_F C_3}{r_0^{2n-2}} + C_F C_4^n C_5^{n-1}$ such that $\gamma_t(x)$ is bounded by ν from above. On the other hand, we have the lower bound of $\gamma_t \ge C_\rho^{-1}$ by similar computation in [DGG20, bottom of page 23]. Therefore, we obtain a two-sided bound of γ_t:

$$
\forall x \in \mathbb{B}_{1/2} \text{ and } \forall t \in \overline{\mathbb{D}}_r, \quad C_\rho^{-1} \le \gamma_t(x) \le \nu. \tag{4.3.2}
$$

Step 3: Conclusion. We closely follow the strategy in [DGG20, page 24-25] to conclude. Combining (4.3.1), (4.3.2) and Jensen's inequality, we derive

$$
e^{-\alpha I_t(x)} = \exp\left(\int_{z \in \mathbb{B}} -\alpha \gamma_t(x) G_x d\mu_t \right) \le C_\rho \int_{z \in \mathbb{B}} \frac{dd^c v_t \wedge (dd^c |z|^2)^{n-1} \wedge [X_t]}{|\Phi_x(z)|^{\alpha\nu + 2n - 2}}.
$$

Integrating $x \in \mathbb{B}_{1/2}$ and using Fubini's theorem, one can infer the following inequality

$$\begin{aligned}&\int_{x\in\mathbb{B}_{1/2}} e^{-\alpha u_t}(dd^c|x|^2)^n \wedge [X_t] \leq \int_{x\in\mathbb{B}_{1/2}} e^{-\alpha v_t}(dd^c|x|^2)^n \wedge [X_t]\\ &\leq e^{\alpha C_F C_1} C_\rho \int_{z\in\mathbb{B}} \left(\int_{x\in\mathbb{B}_{1/2}} \frac{(dd^c|x|^2)^n \wedge [X_t]}{|\Phi_x(z)|^{\alpha\nu+2n-2}}\right) dd^c v_t(z) \wedge (dd^c|z|^2)^{n-1} \wedge [X_t]\end{aligned}$$

Fix a constant α sufficiently small such that $\alpha\nu < 2$ and put $\beta = \frac{2-\alpha\nu}{2n} > 0$. Following the similar proof of [DGG20, Lemma 2.13], we have

$$C_\beta^{-1}(dd^c_x|\Phi_x(z)|^{2\beta})^n \leq \frac{(dd^c|x|^2)^n}{|\Phi_x(z)|^{\alpha\nu+2n-2}} \leq C_\beta(dd^c_x|\Phi_x(z)|^{2\beta})^n.$$

Now, using the same trick in the proof of Chern–Levine–Nirenberg inequality, one has

$$\begin{aligned}\int_{x\in\mathbb{B}_{1/2}} \frac{(dd^c|x|^2)^n \wedge [X_t]}{|\Phi_x(z)|^{\alpha\nu+2n-2}} &\leq C_\beta \int_{x\in\mathbb{B}_{1/2}} (dd^c_x|\Phi_x(z)|^{2\beta})^n \wedge [X_t]\\ &\leq C_\beta \int_{x\in\operatorname{supp}(\chi_1)} \chi_1 (dd^c_x|\Phi_x(z)|^{2\beta})^n \wedge [X_t]\\ &\leq C_4^n C_\beta \int_{x\in\mathbb{B}} (dd^c|x|^2)^n \wedge [X_t] \leq C_4^n C_\omega^n C_\beta \operatorname{Vol}_{\omega_t}(\Omega_t).\end{aligned}$$

Note that in global Kähler cases, the above estimate is controlled by cohomology classes. Finally, we get the estimate on each $\mathbb{B}_{1/2}$

$$\begin{aligned}\int_{x\in\mathbb{B}_{1/2}} e^{-\alpha u_t}(dd^c|x|^2)^n \wedge [X_t] &\leq e^{\alpha C_F C_1} \int_{z\in\mathbb{B}} (C_4^n C_\omega^n C_\beta \operatorname{Vol}_{\omega_t}(\Omega_t))\, dd^c v_t \wedge (dd^c|z|^2)^{n-1} \wedge [X_t]\\ &\leq e^{\alpha C_F C_1} C_4^n C_\omega^n C_\beta \operatorname{Vol}_{\omega_t}(\Omega_t) \int_{\mathbb{B}_2} \chi dd^c v_t \wedge (dd^c|z|^2) \wedge [X_t]\\ &\leq e^{\alpha C_F C_1} C_4^{n+1} C_\omega^n C_\beta \operatorname{Vol}_{\omega_t}(\Omega_t) \int_{\Omega_t} (-v_t)(dd^c|z|^2)^n\\ &\leq e^{\alpha C_F C_1} C_F C_4^{n+1} C_\omega^n C_\beta \operatorname{Vol}_{\omega_t}(\Omega_t).\end{aligned}$$

Note that $t \mapsto \operatorname{Vol}_{\omega_t}(\Omega_t)$ is continuous. One has a uniform control $\int_{x\in\mathbb{B}_{1/2}} e^{-\alpha u_t}(dd^c|x|^2)^n \wedge [X_t]$ for all t close to 0. Summing the integration on every $\mathbb{B}_{1/2}$ in the collection, we obtain the estimate in Theorem 4.3.1 as desired. □

4.3.2 Global Skoda's estimate

Now, we assume that there is a uniform constant $C_{SL} > 0$ such that X_t satisfies (SL) for all $t \in \mathbb{D}$. As a consequence, we have the following uniform global version of Skoda's estimate:

Proposition 4.3.3. *Assume that there is a uniform constant $C_{SL} > 0$ such that for all $t \in \mathbb{D}$ and for* *every* $u_t \in \mathrm{PSH}(X_t, \omega_t)$ *with* $\sup_{X_t} = 0$,

$$\frac{1}{V_t}\int_{X_t}(-u_t)\omega_t^n \leq C_{SL}$$

where $V_t := \mathrm{Vol}_{\omega_t}(X_t)$. *Then there exists constants* α, A_α *such that for all* $t \in \overline{\mathbb{D}}_{1/2}$ *and for all* $u_t \in \mathrm{PSH}(X_t, \omega_t)$ *with* $\sup_{X_t} u_t = 0$,

$$\int_{X_t} \mathrm{e}^{-\alpha u_t}\omega_t^n \leq A_\alpha.$$

Proof. Without loss of generality, we just treat the proof for t in a small neighborhood near $0 \in \mathbb{D}$. Let $(\mathcal{U}_j)_{j\in J}$ and $(\mathcal{U}'_j)_{j\in J}$ be a strongly pseudoconvex finite double cover of $\pi^{-1}(\mathbb{D}_r)$ for some $r > 0$ sufficiently small. We write $\mathcal{U}'_j := \{\rho_j < -c_j\} \Subset \mathcal{U}_j = \{\rho_j < 0\}$ for some $c_j > 0$. For simplicity, we may assume that $dd^c\rho_j \geq \omega$ for all $j \in J$. Also, we set the slices $\Omega_{t,j} := X_t \cap \mathcal{U}_j$ and $\Omega'_{t,j} := X_t \cap \mathcal{U}'_j$. For all $u_t \in \mathrm{PSH}(X_t, \omega_t)$ with $\sup_{X_t} u_t = 0$, it is obvious that $u_t + \rho_{t,j} \in \mathrm{PSH}(\Omega_{t,j})$. Note that for all $j \in J$,

$$\int_{\Omega_{t,j}} -(u_t + \rho_{t,j})\omega_t^n \leq \int_{X_t} -u_t\omega_t^n + V_t \left\|\rho_j\right\|_{L^\infty(\mathcal{U}_j)} \leq C_F$$

for some uniform constant C_F. By Theorem 4.3.1, we obtain

$$\forall j \in J, \quad \int_{\Omega'_{t,j}} \mathrm{e}^{-\alpha_j u_t}\omega_t^n \leq \int_{\Omega'_{t,j}} \mathrm{e}^{-\alpha_j(u_t+\rho_{t,j})}\omega_t^n \leq A_{\alpha_j,j}.$$

Since J is a finite set of indices, one can easily derive the desired estimate. □

This ensures that one can find uniform geometric constants (Skoda) in a family which fulfills Conjecture (SL).

4.4 Uniform volume-capacity comparison

This section aims to deal with the volume-capacity comparison (VC) in a family $\pi : \mathcal{X} \to \mathbb{D}$ with locally irreducible fibres. Let $\mathcal{U} = \{\rho < 0\}$ be a strongly pseudoconvex domain in $\mathcal{X}$. We also assume that the closure of $\mathcal{U}$ can be contained in a larger strongly pseudoconvex set $\widetilde{\mathcal{U}} = \{\widetilde{\rho} < 0\}$ in $\mathcal{X}$. Denote the slices by $\Omega_t := \mathcal{U} \cap X_t$ and $\widetilde{\Omega}_t := \widetilde{\mathcal{U}} \cap X_t$.

4.4.1 Subextensions and relative extremal functions

We first recall some useful facts in pluripotential theory:

Subextensions

Fix a strongly pseudoconvex domain $\Omega = \{\rho < 0\}$ and a strongly pseudoconvex neighborhood $\widetilde{\Omega} = \{\widetilde{\rho} < 0\}$ containing $\overline{\Omega}$. We define

$$\mathcal{E}^0(\Omega) := \left\{ u \in \mathrm{PSH}(\Omega) \cap L^\infty(\Omega) \,\middle|\, u|_{\partial\Omega} = 0 \text{ and } \int_\Omega (\mathrm{dd^c} u)^n < +\infty \right\}.$$

We recall some properties of subextension of the plurisubharmonic functions in $\mathcal{E}^0(\Omega)$.

Lemma 4.4.1 ([CZ03, Theorem 2.2] and [GGZ23, Lemma 1.7])**.** *Suppose that $\varphi \in \mathcal{E}^0(\Omega)$. The subextension of φ is defined as:*

$$\widetilde{\varphi} := \sup \left\{ u \in \mathrm{PSH}(\widetilde{\Omega}) \,\middle|\, u|_{\partial\widetilde{\Omega}} \leq 0 \textit{ and } u \leq \varphi \textit{ in } \Omega \right\}.$$

Then the subextension $\widetilde{\varphi}$ satisfies the following properties:

(i) $\widetilde{\varphi} \in \mathcal{E}^0(\widetilde{\Omega})$*;*

(ii) $\widetilde{\varphi} \leq \varphi$ *on* Ω*;*

(iii) $\int_{\widetilde{\Omega}} (\mathrm{dd^c} \widetilde{\varphi})^n \leq \int_\Omega (\mathrm{dd^c} \varphi)^n$*.*

Relative extremal functions

Next, we review the definition and some basic properties of relative extremal functions (cf. [GZ17, Chapter 3]):

Definition and Proposition 4.4.2. *Let E be a Borel subset in a strongly pseudoconvex domain Ω. The relative extremal function with respect to (E, Ω) is defined as follows*

$$h_{E;\Omega}(z) = \sup \left\{ u(z) \,|\, u \in \mathrm{PSH}(\Omega),\, u \leq 0 \textit{ and } u|_E \leq -1 \right\}.$$

Suppose that E is a relatively compact Borel subset in Ω. Then we have the following facts:

(i) The function $h^*_{E;\Omega}$ is psh in Ω and $h^*_{E;\Omega} = -1$ on E off a pluripolar subset;

(ii) The boundary value of $h^*_{E;\Omega}$ is zero (i.e. $\lim_{z \to \partial\Omega} h^*_{E;\Omega}(z) = 0$);

(iii) The Monge–Ampère measure $(\mathrm{dd^c} h^*_{E;\Omega})^n$ puts no mass on $\Omega \setminus \overline{E}$;

(iv) The capacity can be expressed by the integration of Monge–Ampère measure of the relative extremal function: $\mathrm{Cap}(E; \Omega) = \int_\Omega (\mathrm{dd^c} h^*_{E;\Omega})^n$*.*

4.4.2 Volume-capacity comparison

Then we explain the volume-capacity comparison in families:

Proposition 4.4.3. *For every $k > 1$, there exists a constant $C_{VC,k}$ such that*

$$\forall K_t \Subset \Omega_t, \quad \mathrm{Vol}(K_t) \le C_{VC,k} \,\mathrm{Cap}^k(K_t; \Omega_t),$$

for all $t \in \mathbb{D}_{1/2}$

In smooth setting, the proof of the volume-capacity comparison was first given by Kołodziej [Koł02]. For global semi-positive setup, Guedj and Zeriahi [GZ05] provided a proof simply using Skoda's integrability theorem and the comparison of Bedfor–Taylor and Alexander–Taylor capacities. For local cases, in [GGZ23, Lemma 1.9], Guedj–Guenancia–Zeriahi gave an easy proof for us to chase the depending constants. We also compute explicitly the constant $C_{VC,k}$ to verify that it is independent of t.

Proof. Without loss of generality, we may assume that K_t is non-pluripolar. Otherwise, both sides of the inequality are zeros. We define

$$u_{K_t} = \frac{h^*_{K_t;\Omega_t}}{\mathrm{Cap}^{1/n}(K_t;\Omega_t)}.$$

According to Proposition 4.4.2, one has $u_{K_t} \in \mathcal{E}^0(\Omega_t)$ and $\int_{\Omega_t} (\mathrm{dd^c} u_{K_t})^n = 1$. Recall that $\widetilde{\mathcal{U}} = \{\widetilde{\rho} < 0\}$ is a strongly pseudoconvex neighborhood of $\overline{\mathcal{U}}$ and $\widetilde{\Omega}_t = \widetilde{\mathcal{U}} \cap X_t$. Let $C_{\widetilde{\rho}} > 0$ be a constant such that $\mathrm{d}V \le C_{\widetilde{\rho}} (\mathrm{dd^c} \widetilde{\rho})^n$ on $\widetilde{\mathcal{U}}$. Consider the subextension of u_{K_t}:

$$\widetilde{u}_{K_t} = \sup \left\{ u \in \mathrm{PSH}(\widetilde{\Omega}_t) \cap L^\infty(\widetilde{\Omega}_t) \;\middle|\; u|_{\partial \widetilde{\Omega}_t} \le 0 \text{ and } u \le u_{K_t} \text{ in } \Omega_t \right\}.$$

By Lemma 4.4.1, we have $\widetilde{u}_{K_t} \in \mathcal{E}^0(\widetilde{\Omega}_t)$, $\widetilde{u}_{K_t} \le u_{K_t}$ in Ω_t, and $\int_{\widetilde{\Omega}_t} (\mathrm{dd^c} \widetilde{u}_{K_t})^n \le \int_{\Omega_t} (\mathrm{dd^c} u_{K_t})^n = 1$. Using the integration by parts and the condition $\int_{\widetilde{\Omega}_t} (\mathrm{dd^c} \widetilde{u}_{K_t})^n \le 1$, one can see that $\|\widetilde{u}_{K_t}\|_{L^1(\widetilde{\Omega}_t)}$ is uniformly bounded independent of K_t. Indeed,

$$\begin{aligned}
\int_{\widetilde{\Omega}_t} (-\widetilde{u}_{K_t}) \mathrm{d}V &\le \mathrm{Vol}^{\frac{n-1}{n}}(\widetilde{\Omega}_t) \left(\int_{\widetilde{\Omega}_t} (-\widetilde{u}_{K_t})^n \mathrm{d}V \right)^{1/n} \\
&\le \mathrm{Vol}^{\frac{n-1}{n}}(\widetilde{\Omega}_t) C_{\widetilde{\rho}} \left(\int_{\widetilde{\Omega}_t} (-\widetilde{u}_{K_t})^n (\mathrm{dd^c} \widetilde{\rho}_t)^n \right)^{1/n} \\
&\le \mathrm{Vol}^{\frac{n-1}{n}}(\widetilde{\Omega}_t) C_{\widetilde{\rho}} \left\| \widetilde{\rho}_t \right\|_{L^\infty(\widetilde{\Omega}_t)} \left(\int_{\widetilde{\Omega}_t} (\mathrm{dd^c} \widetilde{u}_{K_t})^n \right)^{1/n} \le \mathrm{Vol}^{\frac{n-1}{n}}(\widetilde{\Omega}_t) C_{\widetilde{\rho}} \left\| \widetilde{\rho} \right\|_{L^\infty(\widetilde{\mathcal{U}})}.
\end{aligned}$$

According to Theorem 4.3.1, there exists constants $\alpha, A_\alpha > 0$ such that for all $K_t \Subset \Omega_t$ non-pluripolar,

$$\int_{\Omega_t} \mathrm{e}^{-\alpha \widetilde{u}_{K_t}} \mathrm{d}V \le A_\alpha.$$

Recall that $h^*_{K_t} = -1$ on K_t almost everywhere. By the definition of u_{K_t} and $\widetilde{u}_{K_t} \leq u_{K_t}$, we have

$$\mathrm{Vol}(K_t) \cdot \exp\left(\frac{\alpha}{\mathrm{Cap}^{1/n}(K_t;\Omega_t)}\right) = \int_{K_t} e^{-\alpha u_{K_t}} dV \leq \int_{\Omega_t} e^{-\alpha u_{K_t}} dV \leq A_\alpha$$

and this implies

$$\mathrm{Vol}(K_t) \leq A_\alpha \exp\left(-\frac{\alpha}{\mathrm{Cap}^{1/n}(K_t;\Omega_t)}\right) \leq A_\alpha \frac{b_k}{\alpha^{kn}} \mathrm{Cap}^k(K_t;\Omega_t).$$

where b_k is a numerical constant such that $\exp(-1/x) \leq b_k x^{kn}$ for all $x > 0$. □

4.4.3 Global volume-capacity comparison

In this section, we show the uniform volume-capacity comparison in a given family of compact hermitian varieties $\pi : (\mathcal{X}, \omega) \to \mathbb{D}$ with locally irreducible fibres.

Let (X, ω) be a compact hermitian variety. We define the similar concept of Monge–Ampère capacity with respect to ω-psh functions by

$$\mathrm{Cap}_\omega(K) := \sup\left\{\int_K (\omega + dd^c u)^n \,\middle|\, u \in \mathrm{PSH}(X,\omega),\ \text{and } 0 \leq u \leq 1\right\}.$$

On the other hand, fixing a pseudoconvex finite double cover $(\Omega_j)_j$ and $(\Omega'_j)_j$ of X such that $\Omega'_j \Subset \Omega_j$, we define the Bedford–Taylor capacity by

$$\mathrm{Cap}_{\mathrm{BT}}(K) := \sum_j \mathrm{Cap}(K \cap \overline{\Omega'_j}; \Omega_j).$$

To prove the global volume-capacity comparison, we should first compare the Bedford–Taylor capacity and the capacity of ω-psh functions.

Lemma 4.4.4. *There exists a constant $C_{BT,\omega} > 0$ such that*

$$\forall \text{ compact subset } K_t \subset X_t, \quad C^{-1}_{BT,\omega} \mathrm{Cap}_{\mathrm{BT}}(K_t) \leq \mathrm{Cap}_{\omega_t}(K_t) \leq C_{BT,\omega} \mathrm{Cap}_{\mathrm{BT}}(K_t)$$

for all $t \in \overline{\mathbb{D}}_{1/2}$.

Proof. On a fixed compact Kähler manifold, a similar version of Lemma 4.4.4 was provided by Kołodziej [Koł03, Section 1]. The proof of Lemma 4.4.4 is similar to Kołodziej's proof. For the reader's convenience, we include the proof here.

After shrinking $\mathbb{D}$, we may assume that $(\mathcal{U}_j)_j$ and $(\mathcal{U}'_j)_j$ form a pseudoconvex double cover of $\mathcal{X}$ such that $\mathcal{U}_j = \{\rho_j < 0\}$ for some strictly psh function ρ_j and $\mathcal{U}_j = \{\rho_j < -c_j\}$. Multiplying a positive constant, we may assume that $\frac{1}{C} dd^c_{\mathcal{X}} \rho_j \leq \omega \leq C dd^c_{\mathcal{X}} \rho_j$. Let $C' > 0$ be a constant so that $0 \leq \rho_j \leq C'$ for all j.

Write $\Omega_{j,t} = \mathcal{U}_j \cap X_t$ and $\Omega'_{j,t} = \mathcal{U}'_j \cap X_t$. Fix $u_t \in \mathrm{PSH}(X_t, \omega_t)$ with $0 \le u_t \le 1$. Let K_t be a compact subset of X_t and $K_{t,j} := K_t \cap \overline{\Omega'_j}$. Then we have

$$\begin{aligned}\int_{K_t} (\omega_t + \mathrm{dd}^c_t u_t)^n \le& \sum_j \int_{K_{j,t}} (\mathrm{dd}^c_t (C\rho_j + u_t))^n \\ =& \sum_j \int_{K_{j,t}} (CC'+1)^n \left(\mathrm{dd}^c_t \left(\frac{C\rho_j + u_t}{CC'+1} \right) \right)^n \\ \le& (CC'+1)^n \sum_j \mathrm{Cap}(K_{j,t}; \Omega_j) = (CC'+1)^n \,\mathrm{Cap}_{\mathrm{BT}}(K_t)\end{aligned}$$

and hence $\mathrm{Cap}_{\omega_t}(K_t) \le (CC'+1)^n \,\mathrm{Cap}_{\mathrm{BT}}(K_t)$.

On the other hand, we shall use the gluing argument to prove the local capacity is bounded by the global capacity. Suppose $u_t \in \mathrm{PSH}(\Omega_{j,t})$ and $0 \le u_t \le 1$. Consider a smooth function χ_j defined on $\mathcal{U}_j$ such that

$$\chi(z) = \begin{cases} -1 & \text{when } z \in \mathcal{U}'_j \\ 2 & \text{when } z \text{ is in a neighborhood of } \partial \mathcal{U}_j \end{cases}.$$

We can find a small $\delta_j \in (0, \frac{1}{3})$ such that $\delta_j \chi_j$ can be extended to a ω-psh function on $\mathcal{X}$. Now, we us the gluing argument to define a ω_t-psh function ψ and it is identically equal to $\delta_j u_t$ in $\Omega'_{j,t}$

$$\psi(z) = \begin{cases} \delta_j u(z) & \text{when } z \in \Omega'_{j,t} \\ \max\{\delta_j \,\chi_j|_{X_t}, \delta_j u_t\} & \text{when } z \in \Omega_{j,t} \setminus \Omega'_{j,t} \\ \delta_j \chi_j & \text{when } z \in X_t \setminus \Omega_{j,t} \end{cases}.$$

Obviously, $\tilde{\psi} = \psi + 1/3$ is a ω_t-psh function and $0 \le \tilde{\psi} \le 1$. Then, we obtain

$$\int_{K_{j,t}} (\mathrm{dd}^c_t u_t)^n = \frac{1}{\delta_j^n} \int_{K_{j,t}} (\mathrm{dd}^c_t \delta_j u_t)^n \le \frac{1}{\delta_j^n} \int_{K_t} (\omega_t + \mathrm{dd}^c_t \tilde{\psi})^n \le \frac{1}{\delta_j^n} \mathrm{Cap}_{\omega_t}(K_t).$$

□

Combining Proposition 4.4.3 and Lemma 4.4.4, we have the global volume-capacity comparison in families:

Proposition 4.4.5. *Given $k > 1$, there exists a uniform constant $C_{GVC,k} \ge 1$ such that*

$$\forall \text{ compact subset } K_t \subset X_t, \quad \mathrm{Vol}_{\omega_t}(K_t) \le C_{GVC,k} \,\mathrm{Cap}^k_{\omega_t}(K_t)$$

for all $t \in \mathbb{D}_{1/2}$.

4.5 Sup-L^1 comparison in families

In this section, we pay a special attention to Conjecture (SL). We shall follow the strategy of proof in [DGG20, Section 3]. First of all, we recall the assumption which will be used in this section:

Assumption 4.5.1 (=Geometric assumption (GA)). Suppose that $\pi : \mathcal{X} \to \mathbb{D}$ is a family of hermitian varieties satisfies the geometric setting (GS) and one of the following conditions:

(a) π is locally trivial;

(b) $\pi : \mathcal{X} \to \mathbb{D}$ is a smoothing of X_0 and X_0 has only isolated singularities.

Under such assumption, we establish a uniform L^1-estimate of ω_t-psh function:

Proposition 4.5.2 (=Proposition 4.0.2). *Suppose that $\pi : \mathcal{X} \to \mathbb{D}$ satisfies the geometric assumption (GA). Then there exists a uniform constant $C > 0$ such that for all $t \in \mathbb{D}_{1/2}$*

$$\forall \varphi_t \in \mathrm{PSH}(X_t, \omega_t), \quad \sup_{X_t} \varphi_t - C \leq \frac{1}{V_t} \int_{X_t} \varphi_t \omega_t^n \leq \sup_{X_t} \varphi_t,$$

where $V_t := \mathrm{Vol}_{\omega_t}(X_t)$.

4.5.1 Proof of Proposition 4.0.2

Locally trivial families

In locally trivial cases, one can follow the similar strategy in [DGG20, Section 3.2] for the proof. The only difference is to replace the local potentials of ω_0 by the local inequalities $0 < C_\rho^{-1} dd^c \rho \leq \omega_0 \leq C_\rho dd^c \rho$ for some local psh function ρ and some constant $C_\rho > 0$ on X_0.

Smoothing of varieties with isolated singularities

Before diving into the main goal, we recall the uniform boundedness of the integral of Laplacian.

Lemma 4.5.3. *Suppose that $\pi : (\mathcal{X}, \omega) \to \mathbb{D}$ satisfies the geometric setting (GS) and it is a smoothing of a compact variety X_0. There is a uniform constant $C_{\mathrm{Lap}} > 0$ such that*

$$\forall \varphi_t \in \mathrm{PSH}(X_t, \omega_t), \quad \int_{X_t} (\omega_t + dd^c_t \varphi_t) \wedge \omega_t^{n-1} \leq C_{\mathrm{Lap}},$$

for all $t \in \mathbb{D}_{1/2}$.

Proof. Recall that from [Pan22, Theorem A], there is a uniform constant $C_G > 0$ such that for all $t \in \mathbb{D}^*_{1/2}$ the normalized Gauduchon factor g_t with respect to (X_t, ω_t) (i.e. $dd^c_t \left(g_t \omega_t^{n-1}\right) = 0$)

is bounded between 1 and C_G. Then we have

$$\begin{aligned}\int_{X_t}(\omega_t+\mathrm{dd}^c_t\varphi_t)\wedge\omega_t^{n-1} &\leq \int_{X_t}(\omega_t+\mathrm{dd}^c_t\varphi_t)\wedge g_t\omega_t^{n-1}\\ &\leq \int_{X_t} g_t\omega_t^n+\int_{X_t}\varphi_t\mathrm{dd}^c_t(g_t\omega_t^{n-1})\leq C_G\operatorname{Vol}_{\omega_t}(X_t)\end{aligned}$$

for all $t\in\mathbb{D}^*_{1/2}$. Since $(\operatorname{Vol}_{\omega_t}(X_t))_{t\in\mathbb{D}_{1/2}}$ is uniformly bounded from above (cf. [Pan22, Section 1.4]), we have a desired estimate for all $t\in\mathbb{D}^*_{1/2}$.

On the central fibre X_0, there always exists a constant $C_{SL,0}>0$ such that

$$\frac{1}{\operatorname{Vol}_{\omega_0}(X_0)}\int_{X_0}-\varphi_0\omega_0^n\leq C_{SL,0}$$

for all $\varphi_0\in\operatorname{PSH}(X_0,\omega_0)$ with $\sup_{X_0}\varphi_0=0$. For all $\varphi_0\in\operatorname{PSH}(X_0,\omega_0)$, we take $\tilde{\varphi}_0=\varphi_0-\sup_{X_0}\varphi_0$. Then one can use the argument as in Lemma 4.2.7 to find a constant $C'_{\mathrm{Lap}}>0$ such that $\int_{X_0}(\omega_0+\mathrm{dd}^c_0\varphi_0)\wedge\omega_0^{n-1}\leq C'_{\mathrm{Lap}}$. We have thus obtained the desired estimate. □

We follow the idea in [DGG20, Section 3.3] to get the proof. Note that we only need to take care of sequences of sup-normalized ω_{t_k}-psh functions $(\varphi_{t_k})_k$ where $t_k\xrightarrow[k\to+\infty]{}0$.

Step 1: Choose a good covering and a test function. Following the same argument in [DGG20, page 30, Step 2], up to shrinking $\mathbb{D}$, we can find a finite open covering $(\mathcal{V}_i)_{i\in I}$ of $\mathcal{X}$ such that

(i) each point of $\mathcal{Z}:=\mathcal{X}^{\mathrm{sing}}=X_0^{\mathrm{sing}}$ belongs to exactly one element of $\mathcal{V}_i$ of the covering, we denote by J the collection of indices of these open subsets;

(ii) on each $\mathcal{V}_i$, we have a smooth strictly psh function ρ_i such that

$$C_\rho^{-1}\mathrm{dd}^c\rho_i\leq\omega\leq C_\rho\mathrm{dd}^c\rho_i\quad\text{and}\quad 0\leq\rho_i\leq C_\rho$$

for a uniform constant $C_\rho>0$;

(iii) for each $i\in J$, there is a relatively compact open subset $\mathcal{W}_i\Subset\mathcal{V}_i$ with $\mathcal{W}_i\cap\mathcal{Z}\neq\emptyset$.

Define

$$\delta:=\frac{1}{2}\min_{i\in J}\left\{\operatorname{dist}_\omega(\partial\mathcal{W}_i,\mathcal{W}_i\cap X_0^{\mathrm{sing}})\right\}>0.$$

Let χ_i be a cut-off function supported in $\mathcal{V}_i$ and $\chi_i\equiv 1$ in a neighborhood of $\mathcal{W}_i$. Set $\rho=\sum_{i\in I}\chi_i\rho_i$. Obviously, we have $\omega\leq C_\rho\mathrm{dd}^c\rho$ on $\mathcal{W}:=\cup_{i\in I}\mathcal{W}_i$. Furthermore, we may assume $-C_\rho\omega\leq\mathrm{dd}^c\rho\leq C_\rho\omega$ on $\mathcal{X}$ by choosing larger C_ρ.

Step 2: Uniform L^1-estimate away from singularities. Define a set

$$\mathcal{R}:=\left\{p\in\mathcal{X}\,\middle|\,\operatorname{dist}_\omega(p,X_0^{\mathrm{sing}})>\delta/2\right\}.$$

Since $\mathcal{R}^c$ lies in $\mathcal{W}$, after shrinking $\mathbb{D}$, we can cover $\mathcal{R}$ by finitely many open subsets $(\mathcal{U}_i)_i$ away from the singular locus. We may assume that π is locally trivial on $\mathcal{R}$ with respect to $(\mathcal{U}_i)_i$ because π is a submersion on $\mathcal{R}$.

Following the argument in [DGG20, page 31, Step 3], one can prove that there is a constant $C > 0$ and a subsequence of $(t_k)_k$ such that

$$\sup_{\mathcal{R}\cap X_{t_k}} \varphi_{t_k} \geq -C.$$

By the irreducibility of X_0, $\mathcal{R}$ is connected. Then one can use the same proof in locally trivial cases to show that

$$\int_{\mathcal{R}\cap X_{t_k}} (-\varphi_{t_k})\omega_{t_k}^n \leq C_{\mathcal{R}}$$

for some uniform constant $C_{\mathcal{R}} > 0$.

Step 3: Conclusion. Recall that on $\mathcal{W}$ we have $\omega \leq C_\rho dd^c\rho$. Define a smooth (n,n)-form $\Omega := \omega^n - C_\rho^n(dd^c\rho)^n$. It is easy to see that $\Omega|_{\mathcal{W}\cap X_t} \leq 0$ and $\Omega_t := \Omega|_{X_t} \leq C_\Omega \omega_t^n$ for some uniform constant $C_\Omega > 0$. Note that $\mathcal{R}^c \subset \mathcal{W}$. We have

$$\begin{aligned}\int_{X_{t_k}} (-\varphi_{t_k})\Omega_{t_k} &= \int_{\mathcal{R}\cap X_{t_k}} (-\varphi_{t_k})\Omega_{t_k} + \int_{\mathcal{R}^c\cap X_{t_k}} (-\varphi_{t_k})\Omega_{t_k} \\ &\leq C_\Omega \int_{\mathcal{R}\cap X_{t_k}} (-\varphi_{t_k})\omega_{t_k}^n \leq C_{\mathcal{R}} C_\Omega.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\int_{X_{t_k}} (-\varphi_{t_k})(dd^c\rho)^n &= \int_{X_{t_k}} -\rho dd^c\varphi_{t_k} \wedge (dd^c\rho)^{n-1} \\ &= -\int_{X_{t_k}} \rho(\omega_{t_k} + dd^c\varphi_{t_k}) \wedge (dd^c\rho)^{n-1} + \int_{X_{t_k}} \rho\omega_{t_k} \wedge (dd^c\rho)^{n-1} \\ &\leq C_\rho^n \int_{X_{t_k}} (\omega_{t_k} + dd^c\varphi_{t_k}) \wedge \omega_{t_k}^{n-1} + C_\rho^n \operatorname{Vol}_{\omega_{t_k}}(X_{t_k}) \\ &\leq C_\rho^n C_{\mathrm{Lap}} + C_\rho^n \operatorname{Vol}_{\omega_{t_k}}(X_{t_k}).\end{aligned}$$

The fourth line comes directly from Lemma 4.5.3. All in all, we obtain a uniform L^1-estimate of sup-normalized ω_{t_k}-psh functions:

$$\int_{X_{t_k}} (-\varphi_{t_k})\omega_{t_k}^n = C_\rho^n \int_{X_{t_k}} (-\varphi_{t_k})(dd^c\rho)^n + \int_{X_{t_k}} (-\varphi_{t_k})\Omega_{t_k} \leq C_\rho^{2n} C_{\mathrm{Lap}} + C_\rho^{2n} \operatorname{Vol}_{\omega_{t_k}}(X_{t_k}) + C_{\mathcal{R}} C_\Omega.$$

4.6 Families of Calabi–Yau varieties

In this final section, we apply our results to families of Calabi–Yau varieties. By Calabi–Yau variety, we mean a normal variety with canonical singularities and trivial canonical bundle.

4.6.1 Canonical singularities

We first recall the notion of canonical singularities. The reader is referred to [EGZ09, Section 5] for more details. Let X be a normal variety. We say that X has *canonical singularities* if the pluricanonical sheaf $K_X^{[N]}$ is locally free for some $N \in \mathbb{N}$, and for any resolution of singularities $p : \widetilde{X} \to X$, for any local generator α of $K_X^{[N]}$, the meromorphic pluricanonical form $p^*\alpha$ is holomorphic.

Lemma 4.6.1. *Suppose that $\pi : \mathcal{X} \to \mathbb{D}$ is a proper surjective holomorphic map satisfying the geometric setting (GS). In addition, assume that*

- *$\mathcal{X}$ is normal and $K_{\mathcal{X}}$ (or equivalently $K_{\mathcal{X}/\mathbb{D}}$) is locally free;*
- *for each $t \in \mathbb{D}$, X_t has only canonical singularities.*

Then the following are equivalent

- *$K_{\mathcal{X}/\mathbb{D}}$ (or $K_{\mathcal{X}}$) is trivial up to shrinking $\mathbb{D}$;*
- *K_{X_t} is trivial for all t small (i.e. X_t is Calabi–Yau for all t small).*

Proof. We include some arguments here for the reader's convenience. Suppose that $K_{\mathcal{X}/\mathbb{D}}$ (or $K_{\mathcal{X}}$) is trivial. Then we have

$$K_{X_t^{\mathrm{reg}}} \simeq K_{\mathcal{X}^{\mathrm{reg}}/\mathbb{D}}|_{X_t^{\mathrm{reg}}} = (K_{\mathcal{X}^{\mathrm{reg}}} - \pi^* K_{\mathbb{D}})|_{X_t^{\mathrm{reg}}} \simeq \mathcal{O}_{\mathcal{X}^{\mathrm{reg}}}|_{X_t^{\mathrm{reg}}} \simeq \mathcal{O}_{X_t^{\mathrm{reg}}}.$$

When t is close to 0, since X_t is normal and K_{X_t} is reflexive, we have

$$K_{X_t} \simeq (j_t)_*(K_{X_t^{\mathrm{reg}}}) \simeq (j_t)_*(\mathcal{O}_{X_t^{\mathrm{reg}}}) \simeq \mathcal{O}_{X_t}$$

where $j_t : X_t^{\mathrm{reg}} \hookrightarrow X_t$ is the inclusion map for each t.

Now, assume that K_{X_t} is trivial for all t close to 0. Since $(X_t)_{t\in\mathbb{D}}$ are Calabi–Yau varieties, the map $t \mapsto h^0(X_t, K_{X_t})$ is constantly equal to 1. As a direct consequence of Grauert's theorem [Gra60], the direct image sheaf $\pi_*(K_{\mathcal{X}/\mathbb{D}})$ is locally free and

$$\pi_*(K_{\mathcal{X}/\mathbb{D}}) \otimes k(0) \simeq H^0(X_0, K_{X_0}), \tag{4.6.1}$$

where $k(0)$ is the residue field at 0. Let Ω_0 be a nowhere vanishing trivialization section of K_{X_0} on X_0. By the isomorphism (4.6.1), Ω_0 descends to an element s_0 of $\pi_*(K_{\mathcal{X}/\mathbb{D}}) \otimes k(0) = \pi_*(K_{\mathcal{X}/\mathbb{D}})|_0$. Since every line bundle over $\mathbb{D}$ is trivial, we can extend the vector $s_0 \in \pi_*(K_{\mathcal{X}/\mathbb{D}})|_0$ to a non-vanishing section s of $\pi_*(K_{\mathcal{X}/\mathbb{D}})$ after shrinking $\mathbb{D}$. Then s is identified with a section Ω of $K_{\mathcal{X}/\mathbb{D}}$ which has the relation $\Omega|_{X_0} = \Omega_0$. Since Ω_0 is nowhere vanishing, the section Ω is nowhere zero on a neighborhood of the central fibre. □

Remark 4.6.2. One can prove that under the assumption X_0 has canonical singularities, then $\mathcal{X}$ is normal, $\mathbb{Q}$-Gorenstein in a neighborhood of X_0 and moreover X_t has canonical singularities for t close to 0. However, we will not use that result.

4.6.2 Families of Calabi–Yau varieties

It is thus legitimate to work in the following setting:

Setup (CY). *Let $\pi : (\mathcal{X}, \omega) \to \mathbb{D}$ be a family of compact hermitian varieties satisfying the geometric setting (GS). Suppose that*

(i) $\mathcal{X}$ is normal and $K_{\mathcal{X}}$ is trivial;

(ii) for all $t \in \mathbb{D}$, X_t has only canonical singularities.

From Lemma 4.6.1 and the inversion of adjunction (cf. [KM98, Theorem 5.50]), Setting (CY) implies following properties:

(1) $\mathcal{X}$ has canonical singularities;

(2) For all $t \in \mathbb{D}$, K_{X_t} is trivial.

Let Ω be a trivializing section of $K_{\mathcal{X}/\mathbb{D}}$. Define the function γ_t on X_t by the equation

$$\Omega_t \wedge \overline{\Omega_t} = \mathrm{e}^{-\gamma_t} \omega_t^n$$

and γ_t also induces a function γ on $\mathcal{X}$ near X_0. From [DGG20, Lemma 4.4], we have the following uniform integrability property:

Proposition 4.6.3. *Up to shrinking $\mathbb{D}$, there exists $p > 1$ and $C > 0$ such that for all $t \in \mathbb{D}$, we have*

$$\int_{X_t} \mathrm{e}^{-p\gamma_t} \omega_t^n < C.$$

On the other hand, if $K \Subset \mathcal{X}^{\mathrm{reg}}$, then $\gamma \in \mathcal{C}^0(K)$ and it implies that

$$\int_{X_t} \mathrm{e}^{-\gamma_t/n} \omega_t^n \geq \mathrm{e}^{-\frac{\sup_K \gamma_t}{n}} \int_{K_t} \omega_t^n > c$$

for some constant $c > 0$ independent of t close to 0. Therefore the canonical densities satisfy the integral bound (IB) in Theorem 4.0.1.

We are now ready to establish a uniform control of Chern–Ricci flat potentials in a family of Calabi–Yau varieties satisfying Setting (CY) and the geometric assumption (GA).

Proof of Theorem 4.0.3. For all $t \in \mathbb{D}$, K_{X_t} is trivial and $K_{X_t} = K_{\mathcal{X}/\mathbb{D}}|_{X_t}$ as well. Therefore, one can find a non-vanishing section Ω_t of K_{X_t} satisfying $\Omega_t = \Omega|_{X_t}$. According to a recent work of Guedj and Lu [GL21, Theorem E], for each $t \in \mathbb{D}$, there exists a solution $(\varphi_t, c_t) \in (\mathrm{PSH}(X_t, \omega_t) \cap L^\infty(X_t)) \times \mathbb{R}_{>0}$ which solves the complex Monge–Ampère equation

$$(\omega_t + dd^c_t \varphi_t)^n = c_t \Omega_t \wedge \overline{\Omega_t}, \quad \text{and} \quad \sup_{X_t} \varphi_t = 0.$$

According to Theorem 4.0.1, Proposition 4.0.2, and Proposition 4.6.3, there is a uniform constant C_{MA} such that for all $t \in \mathbb{D}_{1/2}$, one has

$$c_t + c_t^{-1} + \|\varphi_t\|_{L^\infty} \leq C_{\mathrm{MA}}$$

and this complete the proof of Theorem 4.0.3. □

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