REGULARITY OF GEODESICS IN THE SPACE OF KÄHLER METRICS

by

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Abstract. — This master thesis is a survey article on the recent progress of Chen [**Che00b**], Błocki [**Bł12**], Boucksom [**Bou12**], Chu-Tosatti-Weinkove [**CTW17**], and Chu-McCleerey [**CM19**] in understanding the regularity problem of geodesics on the space of Kähler metrics on the smooth and some mildly singular cases. The origional problem is proposed by Mabuchi [**Mab87**] and reduced by Semmes [**Sem92**] and Donaldson [**Don99**] in to a totally degenerated complex Monge-Ampère equation on a compact Kähler manifold with boundary.

Contents

1. Introduction

1.1. Problem explanation. — In complex geometry, constructing canonical metrics on compact Kähler manifolds is a central problem. Yau's resolution of the Calabi conjecture [**Yau78**] and the resolution of the Yau-Tian-Donaldson conjecture by Chen-Donaldson-Sun [**CDS15**] are some of the landmark results obtained on the Kähler-Einstein problem. These have been as well several progress in constructing a constant scalar curvature Kähler (cscK) metrics.

In order to investigate canonical metrics, one idea is to consider functionals on the space of Kähler potentials and try to construct canonical metrics as critical points of these functionals. Unfortunately, such functionals are usually not convex along Euclidean segments. In 1987, Mabuchi [Mab87] introduced a new Riemannian structure on the space of Kähler potentials and gave evidence that convexity should hold in this geometry.

Since the space of Kähler potentials is an infinite dimensional Fréchet manifold, one cannot apply Cauchy-Lipschitz theory to guarantee the existence of Mabuchi geodesics. Showing the existence of $\mathcal{C}^{1,1}$-smooth Mabuchi geodesics is the main purpose of this master thesis.

1.2. Historical results. —

(i) In 1987, Mabuchi [Mab87] introduced the following new Riemannian structure on the space of Kähler potentials

$$\langle u, v \rangle_\varphi := \int_X u \cdot v \, \mathrm{MA}_\omega(\varphi).$$

(ii) In 1990s, Donaldson and Semmes [Sem92, Don99] independently showed that the geodesic problem is equivalent to a Dirichlet problem for the homogeneous complex Monge-Ampère equation on $X \times A$,

$$\begin{cases} (\pi^*\omega + dd^c_{X\times A}\Phi)^{n+1} = 0, \\ \Phi(x,0) = \varphi_0(x), \\ \Phi(x,1) = \varphi_1(x), \end{cases} \tag{1.1}$$

where $A \subset \mathbb{C}$ is an annulus and $\Phi(x,t) = \Phi(x,t,s) = \varphi_t$ is a function on $X \times A$ (where $e^{t+is} \in A$) associated with Mabuchi geodesic φ_t connecting two Kähler potential φ_0 and φ_1 and such that $\omega + dd^c_X\Phi > 0$ on each X-slice.

(iii) In 2000, Chen [Che00b] proved the $\mathcal{C}^{1,\bar{1}}$-estimate for the solution. The notation $\mathcal{C}^{1,\bar{1}}$ is the space of $\mathcal{C}^1$ functions with bounded Laplacian.

(iv) Around 2010 Darvas-Lempert-Vivas [DL12, LV13, Dar14] showed there is no $\mathcal{C}^2$-solution in general.

(v) In 2017, Chu-Tosatti-Weinkove [CTW17] completely proved an a priori $\mathcal{C}^{1,1}$-estimate, hence $\mathcal{C}^{1,1}$-regularity is essentially optimal.

(vi) In 2019, Chu-McCleerey [CM19] established a similar result when the underlying variety X is mildly singular.

1.3. Structure of this article. — In Section 2, we explain in more detail the geometric motivation, focusing on Kähler-Einstein metrics and constant scalar curvature Kähler (cscK) metrics. In Section 3, we establish a technical theorem which concatenates works of a lot of experts. We briefly explain the strategy to prove main theorem and show the basic $\mathcal{C}^0$-estimates. Then, we will apply the main theorem to obtain the $\mathcal{C}^{1,1}$-estimate as desired. Section 4, 5, 6 are the most technical a priori estimates in this thesis. We will use the estimate already showed in Section 3 to prove $\mathcal{C}^1$ and $\mathcal{C}^2$-estimates. In Section 7, we briefly explain why we cannot expect that the regularity is better than $\mathcal{C}^{1,1}$ and introduce the mildly singular setting.

Acknowledgement. — The author is grateful to his advisor Vincent Guedj for his supports, suggestions and encouragements during the tough covid lockdown. The author would like to thank Eleonora Di Nezza for the helpful discussion.

2. Geometric motivation

In this section, we introduce some geometric motivation of the main problem. It is a significant problem to find a canonical metric which is the "best" representative of a given Kähler class and satisfies some geometric condition such as Einstein equation and constant scalar curvature metrics.

First of all, we are going to set the notations and conventions. Let X be a compact complex manifold, let ω be a Kähler form on X and $\alpha := [\omega] \in H^{1,1}_{\mathrm{dR}}(X, \mathbb{R})$ be the corresponding de Rham class. Define $\mathcal{K}_\alpha := \{\tilde{\omega} \in \alpha \,|\, \tilde{\omega} \text{ is Kähler}\}$ as the space of Kähler metrics. Also, we can consider

$$\mathcal{K}_\omega := \{\varphi \in \mathcal{C}^\infty(X, \mathbb{R}) \,|\, \tilde{\omega} = \omega + \mathrm{dd}^c\varphi > 0\}.$$

Evidently, we have the following map

$$\begin{aligned} \mathcal{K}_\omega &\longrightarrow \mathcal{K}_\alpha \\ \varphi &\longmapsto \omega_\varphi := \omega + \mathrm{dd}^c\varphi \end{aligned}.$$

According to the well-known $\partial\bar{\partial}$-lemma in Kähler geometry and the maximum principle, this map is surjective and is injective up to adding constants. The space of Kähler potentials $\mathcal{K}_\omega$ can be viewed as an infinite dimensional Fréchet manifold. Indeed, $\mathcal{K}_\omega$ is an convex open subset of the Fréchet vector space $\mathcal{C}^\infty(X)$ and hence the tangent bundle is $T\mathcal{K}_\omega = \mathcal{K}_\omega \times \mathcal{C}^\infty(X)$.

Now, we are going to introduce some functionals and geometries on the space of Kähler potentials $\mathcal{K}_\omega$. Consider the energy functional as the following

$$E(\varphi) := \frac{1}{(n+1)V} \sum_{j=0}^{n} \int_M \varphi\omega_\varphi^j \wedge \omega^{n-j}$$

where $V = \int_M \omega^n$ is the volume with respect to ω. We have the following proposition on E by simple calculus of variation.

Proposition 2.1 **([GZ17, p.271]).** — *We have the following properties:*

(i) The functional E is a primitive of the complex Monge-Ampère operator

$$\frac{\mathrm{d}E(\varphi_t)}{\mathrm{d}t} = \frac{1}{V}\int_X \dot{\varphi}_t(\omega + \mathrm{dd}^c\dot{\varphi}_t)^n = \int_X \dot{\varphi}_t \,\mathrm{MA}(\varphi_t) \tag{2.1}$$

and

$$\frac{\mathrm{d}^2E(\varphi_t)}{\mathrm{d}t^2} = \int_X \ddot{\varphi}_t \,\mathrm{MA}(\varphi_t) - \frac{n}{V}\int_X \mathrm{d}\varphi_t \wedge \mathrm{d}^c\varphi_t \wedge (\omega + \mathrm{dd}^c\varphi_t)^{n-1}. \tag{2.2}$$

(ii) The functional E is concave in Euclidean geometry (i.e. the geodesic starting form φ is $\varphi_t = \varphi + tv$), increasing, satisfies $E(\varphi + c) = E(\varphi) + c$ for all $c \in \mathbb{R}$, $\varphi \in \mathcal{K}_\omega$ and the cocycle condition

$$E(\varphi) - E(\psi) = \frac{1}{(n+1)V} \sum_{j=0}^{n} \int_X (\varphi - \psi)(\omega + dd^c\varphi)^j \wedge (\omega + dd^c\psi)^{n-j}.$$

2.1. Kähler-Einstein metric on Fano manifolds. — Let us recall the usual problem of Kähler-Einstein metrics: given a Kähler form ω, people aim to find another metric $\tilde{\omega} \in [\omega]$ such that the Ricci form $\mathrm{Ric}(\tilde{\omega})$ is proportional to the Kähler form $\tilde{\omega}$. Via $\partial\bar{\partial}$-lemma, the Kähler-Einstein condition:

$$\mathrm{Ric}(\omega) = \lambda\omega$$

can be translated into the corresponding complex Monge-Ampère equations. Note that we mainly concern with $\lambda = \pm 1, 0$, because the Einstein factor can be normalized after rescaling the metric ($c_1(X) < 0 \implies \lambda = -1$, $c_1(X) = 0 \implies \lambda = 0$, and $c_1(X) > 0 \implies \lambda = 1$). First, we recall the corresponding complex Monge-Ampère equation to these three cases:

$$\mathrm{MA}(\varphi) = e^{-\lambda\varphi}\mu \tag{MA_λ}$$

where μ is a smooth volume form with volume 1. For $c_1(X) < 0$ (resp. $c_1(X) = 0$), the problem is solved by Aubin and Yau [Aub76], [Yau78] (resp. Yau [Yau78]). For $c_1(X) > 0$ (Fano manifolds), Chen-Donaldson-Sun [CDS15] proved that there exists a Kähler-Einstein metric on the Fano manifold if and only if it is K-stable (in the sense of geometric invariant theory).

We introduce the Ding functional $\mathcal{F}$ mainly cared and some variational formulas of $\mathcal{F}$.

***Definition 2.1.* —** Given $\lambda \in \mathbb{R}$, consider the functional $\mathcal{F}_\lambda : \mathcal{K}_\omega \to \mathbb{R}$ defined by

$$\mathcal{F}_\lambda(\varphi) := E(\varphi) + \frac{1}{\lambda} \log\left(\int_X e^{-\lambda\varphi} d\mu\right)$$

where μ is a smooth volume form with volume 1. The limit case $\lambda = 0$ corresponds to

$$\mathcal{F}_0(\varphi) = E(\varphi) - \int_X \varphi d\mu = \lim_{\lambda \to 0} \mathcal{F}_\lambda(\varphi).$$

***Proposition 2.2* ([GZ17, p.291]). —** *The following holds:*

$$\frac{d\mathcal{F}_\lambda(\varphi_t)}{dt} = \int_X \dot{\varphi}_t \mathrm{MA}(\varphi_t) - \int_X \frac{\dot{\varphi}_t e^{-\lambda\varphi_t}}{\int_X e^{-\lambda\varphi_t} d\mu} d\mu. \tag{2.3}$$

and

$$\frac{d^2\mathcal{F}_\lambda(\varphi_t)}{dt^2} = \int_X \ddot{\varphi}_t \mathrm{MA}(\varphi_t) - \frac{n}{V} \int_X d\dot{\varphi}_t \wedge d^c\dot{\varphi}_t \wedge (\omega + dd^c\varphi_t)^{n-1}$$
$$- \int_X \frac{\ddot{\varphi}_t e^{-\lambda\varphi_t}}{\int_X e^{-\lambda\varphi_t} d\mu} d\mu + \lambda \left\{ \int_X \frac{(\dot{\varphi}_t)^2 e^{-\lambda\varphi_t}}{\int_X e^{-\lambda\varphi_t} d\mu} d\mu + \left(\int_X \frac{\dot{\varphi}_t e^{-\lambda\varphi_t}}{\int_X e^{-\lambda\varphi_t} d\mu} d\mu \right)^2 \right\}. \tag{2.4}$$

It follows from (2.3) that a critical point of $\mathcal{F}_\lambda$ induces a solution of (MA$_\lambda$). Indeed, when $\lambda = 0$, we have $\mathrm{MA}(\varphi) = \mu$. For $\lambda \neq 0$, $\mathrm{MA}(\varphi) = \frac{e^{-\lambda\varphi\mu}}{\int_X e^{-\lambda\varphi}\mathrm{d}\mu}$ and this induces a new function

$$\tilde{\varphi} = \varphi + \frac{1}{\lambda}\log\left(\int_X e^{-\lambda\varphi}\mathrm{d}\mu\right),$$

which is a solution of (MA$_\lambda$). The functional $\mathcal{F}_\lambda$ is strictly concave for $\lambda = -1$ and is concave for $\lambda = 0$ in Euclidean geometry on $\mathcal{K}_\omega$ (i.e. $\varphi_t = \varphi + tv$ is the geodesic in $\mathcal{K}_\omega$). These implies that the Kähler-Einstein metric for $c_1(X) < 0$ case is unique, and the Ricci flat metric for $c_1(X) = 0$ case is unique up to adding constants. Unfortunately, when $c_1(X) > 0$ (i.e. $\lambda = +1$), the energy part is concave in Euclidean geometry, but the entropy term is convex.

Mabuchi [Mab87] introduced a new geometry on $\mathcal{K}_\omega$ as follows:

Definition 2.2. — The Mabuchi metric at $\varphi \in \mathcal{K}_\omega$ is the L^2-inner product with respect to ω_φ. Explicitly, it is defined by

$$\langle f, g\rangle_{\omega_\varphi} := \int_X fg\omega_\varphi^n.$$

Considering a path φ_t in $\mathcal{K}_\omega$, the length induced by the Mabuchi metric is

$$\ell(\varphi_t) = \int_0^1 \left(\sqrt{\int_X \dot{\varphi}_t{}^2\omega_{\varphi_t}^n}\right)\mathrm{d}t$$

Then, under a variation of $\varphi_t + s\psi_t$ where ψ_t is zero at $t = 0$ and $t = 1$, we obtain the geodesic equation which is

$$\ddot{\varphi}_t - \frac{1}{2}\left|\nabla\dot{\varphi}_t\right|^2_{\omega_{\varphi_t}} = 0. \tag{2.5}$$

Furthermore, the geodesic equation yields that a covariant derivative along the path should be

$$D_t\psi_t = \frac{\mathrm{d}\psi_t}{\mathrm{d}t} - \frac{1}{2}\left\langle\nabla\psi_t, \nabla\dot{\varphi}_t\right\rangle_{\omega_{\varphi_t}}.$$

This connection is a torsion free metric connection with respect to the Mabuchi metric (one can see [Che00b] for more details). By plugging in the geodesic equation into the second variational formula of E, we obtain:

Proposition 2.3. — *E is affine along Mabuchi geodesics.*

Let Ω be an domain in $\mathbb{C}$. Consider curves of metrics φ_t on $-K_X$, where $t \in \Omega$.

Theorem 2.4 **([Ber15]).** — *Assume that $-K_X \geq 0$ in the sense that it has a smooth metric of semi-positive curvature. Let φ_t be a curve of metrics on $-K_X$ such that*

$$\mathrm{dd}^c_{X,t}\varphi_t \geq 0$$

in the sense of currents. Then $t \mapsto -\log\left(\int_X e^{-\varphi_t}\mathrm{d}\mu\right)$ is subharmonic in Ω. In particular, if φ_t does not depend on the imaginary part of t, then the function $t \mapsto -\log\left(\int_X e^{-\varphi_t}\mathrm{d}\mu\right)$ is convex.

Furthermore, in [Ber15], Berndtsson also proved the Kähler-Einstein metric is unique up to some elements in $\mathrm{Aut}_0(X)$ where $\mathrm{Aut}_0(X)$ is the connected component of

$$\mathrm{Aut}(X) := \{f : X \to X \,|\, f \text{ is a biholomorphism}\}$$

containing Id_X. In other words, if ω_0 and ω_1 are both Kähler-Einstein metrics on a Fano manifold X, then $\omega_1 = f^*\omega_0$ for some $f \in \mathrm{Aut}_0(X)$.

Theorem 2.5 **([Ber15]).** — *Assume that $H^{0,1}(X) = 0$, and that the curve of metrics φ_t is independent of the imaginary part of t. Moreover, assume that the metrics φ_t are uniformly bounded in the sense that for some smooth metric on $-K_X$, ψ, $|\varphi_t - \psi| \leq C$. Then, if the function $t \to -\log\left(\int_X e^{-\varphi_t} \mathrm{d}\mu\right)$ is affine in Ω, there is a holomorphic vector field $V \in H^0(X, TX)$ generating a one parameter family of biholomorphisms f_t such that*

$$f_t^*(\omega_t) = \omega_0.$$

Note that for Fano manifolds, $H^{0,1}(X) = 0$ is a necessary condition. Indeed, if $-K_X$ is positive, then $H^{0,1}(X) \simeq \check{H}^1(X, \mathcal{O}) \simeq \check{H}^{n-1}(X, \mathcal{O}(K_X))^* = 0$ by the Kodaira-Serre duality and the Kodaira vanishing theorem. One can check that the geodesic satisfies all the assumptions in Theorem 2.4 and Theorem 2.5.

The previous discussion shows that Mabuchi geodesics play an important role in studying canonical metrics. Since Cauchy-Lipschitz theorem does not apply in Fréchet spaces, the existence is not clear at all.

Problem **(Boundary Value Problem).** — *Given two arbitrary Kähler metrics ω_0 and ω_1, can we find a Mabuchi geodesic φ_t in $\mathcal{K}_\omega$ such that $\omega_0 = \omega + \mathrm{dd}^c\varphi_0$ and $\omega_1 = \omega + \mathrm{dd}^c\varphi_1$?*

Semmes [Sem92] and Donaldson [Don99] independently observed this problem can be reformulated into a totally degenerated complex Monge-Ampère equation. Consider an annulus $A := \{\tau \in \mathbb{C} \,|\, 1 < |\tau| < e\}$ and write $t = \log|\tau|$. Then, we define a new function $\Phi(z, t, s) = \Phi(z, t) = \varphi_t(z)$ for $z \in X$, and $e^{t+is} = \tau \in A$. By simple tensor calculus, we then have the following proposition.

Proposition 2.6 **([Sem92, Don99]).** — *The Mabuchi geodesic equation (2.5) connecting two Kähler potential φ_0 and φ_1 is equivalent to a totally degenerated complex Monge-Ampère equation*

$$\begin{cases} (\pi^*\omega + \mathrm{dd}^c_{X\times A}\Phi)^{n+1} = 0, \\ \Phi(x, 0) = \varphi_0(x), \\ \Phi(x, 1) = \varphi_1(x). \end{cases} \tag{2.6}$$

It turns out that this problem does not admit any smooth solution in general. We shall explain a counter-example of Darvas-Lempert-Vivas [DL12, LV13, Dar14] in Section 7. We will nevertheless prove the existence of a $\mathcal{C}^{1,1}$-smooth solution.

One can more generally consider a compact complex manifold M with non-empty boundary ∂M. Let ω be a Kähler form and $F : M \to \mathbb{R}$ be smooth. We will try and solve the complex Monge-Ampère equation

$$(\omega + \mathrm{dd}^c\varphi)^n = e^F\omega^n, \tag{2.7}$$

where $\varphi \in \mathcal{C}^\infty(M, \mathbb{R}) \cap \mathrm{PSH}(M, \omega)$. We discuss this problem in next section.

2.2. Constant scalar curvature Kähler (cscK) metrics. — Constant curvature problem is an important issue in differential geometry. In dimension 2 cases, it is just the Poincaré uniformization theorem. Generally, studying constant scalar curvature metric in a conformal class on compact Riemannian manifold is the celebrated Yamabe problem which is completely solved by Richard Schoen for all dimension greater than 3. Unfortunately, if one follows the strategy of conformal transformation, the Kähler condition may be destroyed.

In Kähler geometry, Mabuchi introduced a functional $\mathcal{M}$ on $\mathcal{K}_\omega$ such that the critical point of $\mathcal{M}$ is a constant scalar curvature Kähler metric (cscK metric). The Mabuchi geometry also plays an important role in such issues. Before discussing the functional associated to scalar curvature, we introduce the variational formula of the scalar curvature along a curve φ_t.

Proposition 2.7 **([Szé14]). —** *The first variational formula of the scalar curvature is*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}S(\omega_{\varphi_t}) &= -\mathcal{D}^*_{\varphi_t}\mathcal{D}_{\varphi_t}\dot{\varphi}_t + g^{\bar{k}j}(t)\nabla_j S(\omega_{\varphi_t})\nabla_{\bar{k}}\dot{\varphi}_t \\ &= -\overline{\mathcal{D}^*_{\varphi_t}\mathcal{D}_{\varphi_t}\dot{\varphi}_t} + g^{\bar{k}j}\nabla_j\dot{\varphi}_t\nabla_{\bar{k}}S(\omega_{\varphi_t})\end{aligned} \tag{2.8}$$

where $\mathcal{D}_{\varphi_t} : \mathcal{C}^\infty(X,\mathbb{C}) \to \mathcal{C}^\infty(\Omega^{0,1}X \otimes \Omega^{0,1}X)$ *defined by*

$$\mathcal{D}_{\varphi_t} f := \left(\nabla^{\omega_{\varphi_t}}_{\bar{k}}\nabla^{\omega_{\varphi_t}}_{\bar{q}} f\right)\mathrm{d}\bar{z}^k \otimes \mathrm{d}\bar{z}^q.$$

Now, we consider a one form

$$\alpha_\varphi(v) = \int_X v(\bar{S} - S(\omega_\varphi))\omega^n_\varphi$$

on $\mathcal{K}_\omega$, where $\bar{S} = \int_X S(\tilde{\omega})\tilde{\omega}^n$ is the average of scalar curvature for any $\tilde{\omega} \in [\omega]$. Note that $\bar{S}$ is independent to the choice of $\tilde{\omega}$. Via some tensor calculus, one can check that α is closed, namely $\frac{\mathrm{d}}{\mathrm{d}t}\alpha_{\varphi+t\psi(v)}\Big|_{t=0} = 0$. Since $\mathcal{K}_\omega$ is contractible and α is closed, there is a functional $\mathcal{M}$ on $\mathcal{K}_\omega$ such that $\mathrm{d}\mathcal{M} = \alpha$ and this is called the Mabuchi functional.

Definition 2.3 **([Mab87]). —** The Mabuchi functional $\mathcal{M} : \mathcal{K}_\omega \to \mathbb{R}$ is defined by

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}(\varphi_t) = \int_X \dot{\varphi}_t(\bar{S} - S(\omega_{\varphi_t}))\omega^n_{\varphi_t}.$$

By definition a critical point of $\mathcal{M}$ gives a constant scalar curvature metric. Chen [Che00a] showed the following closed formula for $\mathcal{M}$:

$$\mathcal{M}(\varphi) = J(\varphi) + \bar{S}I(\varphi) + \int_X \log\left(\frac{\omega^n_{\varphi_t}}{\omega^n}\right)\omega^n,$$

where

$$J(\varphi) := -\sum_{j=0}^{n-1}\frac{1}{(j+1)!(n-j-1)!}\int_X \varphi\,\mathrm{Ric}(\omega) \wedge \omega^{n-j-1} \wedge (\mathrm{dd}^c\varphi)^j$$

and

$$I(\varphi) := \sum_{j=0}^{n}\frac{1}{(j+1)!(n-j)!}\int_X \varphi\omega^{n-j} \wedge (\mathrm{dd}^c\varphi)^j.$$

Proposition 2.8 **([Mab87]). —** *If smooth Mabuchi geodesics exist, the Mabuchi functional $\mathcal{M}$ is convex along them. Indeed, the second variational formula of $\mathcal{M}$ is*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}(\varphi_t)$$
$$= \int_X \left(\ddot{\varphi}_t(\bar{S} - S(\omega_{\varphi_t}) + \dot{\varphi}_t \left[\mathcal{D}^*_{\varphi_t}\mathcal{D}_{\varphi_t}\dot{\varphi}_t - g^{\bar{k}j}\partial_j S(\omega_{\varphi_t})\partial_{\bar{k}}\dot{\varphi}_t)\right] + \dot{\varphi}_t(\bar{S} - S(\omega_{\varphi_t}))\Delta_{\omega_{\varphi_t}}\dot{\varphi}_t\right)\omega^n_{\varphi_t}.$$

Then, plugging in the geodesic equation (2.5), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}(\varphi_t) = \int_X |\mathcal{D}_{\varphi_t}\dot{\varphi}_t|^2_{\omega_{\varphi_t}}\,\omega^n_{\varphi_t} \geq 0.$$

Unfortunately, smooth Mabuchi geodesics rarely exist.

In [BB17], Berman and Berndtssom proved the Mabuchi functional is convex along "weak geodesics" (a term which we shall explain in Section 3). Besides, Berman and Berndtssom also showed how the convexity of $\mathcal{M}$ yields the uniqueness of cscK metrics up to some element in the identity component of the group of biholomorphisms.

Theorem 2.9 **([BB17]). —** *For any Kähler class $[\omega]$ the Mabuchi functional $\mathcal{M}$ is convex along weak geodesic φ_t connecting any two points φ_0 and φ_1 in $\mathcal{K}_\omega$.*

Theorem 2.10 **([BB17]). —** *Given two cohomologous Kähler metrics ω_0, ω_1 on X with constant scalar curvature, there exists an element f in $\mathrm{Aut}_0(X)$ such that $\omega_0 = f^*\omega_1$.*

3. Main Theorem

3.1. Statement of main estimates and applications. — Before stating the main theorem, we introduce some terminology.

***Definition 3.1.* —**

(i) X is said to be a **complex manifold with boundary** if it is a smooth manifold with boundary and it is endowed coordinate system

$$U_j \simeq \{z \in B_R \,|\, r_j(z) \leq 0\}$$

where B is a ball in $\mathbb{C}^n$ and r_j is a locally defining function of the boundary, namely $B_R \cap \partial X = \{r_j = 0\}$ and $B_R \cap X = \{r_j < 0\}$.

(ii) The **holomorphic tangent bundle of** ∂X, $T^h_{\partial X}$, is the biggest complex subbundle of TX containing $T(\partial X)$, in other words $T^h_{\partial X} = T(\partial X) \cap JT(\partial X)$ where $J \in \mathrm{End}(TX)$ is the complex structure of X.

(iii) The **Levi form** $L_{\partial X,\nu}$ is a Hermitian form on $T^h_{\partial X}$ defined by

$$L_{\partial X,\nu} = \frac{1}{\nu \cdot r}\,\mathrm{d}\mathrm{d}^c r|_{T^h_{\partial X}}$$

where ν is the outward pointing unit normal vector field to ∂X with respect to a given metric ω.

(iv) ∂X is called **weakly (strictly) pseudo-concave (resp. weakly (strictly) pseudo-convex)** if $L_{\partial X,\nu} \leq 0 (< 0)$ (resp. $L_{\partial X,\nu} \geq 0 (> 0)$).

The case which we mainly concern in this section is a compact Kähler manifold (X, ω) with boundary and $\dim_{\mathbb{C}} X = n$. We are going to investigate the following equation

$$(\omega + dd^c\varphi)^n = e^F \omega^n, \tag{3.1}$$

where φ is ω-psh, $\varphi|_{\partial X} = 0$ and F is a smooth function such that

(i) $-A_0 \leq F \leq 0$;
(ii) $\sup_X |\nabla F|_\omega \leq A_1$;
(iii) $\Delta_\omega F \geq -A_2$.

The main theorem provides a priori estimates, concatenating lots of works by Caffarelli-Kohn-Nirenberg-Spruck [CKNS85], Guan [Gua98], Chen [Che00b], Błocki [Bł12], Boucksom [Bou12], and Chu-Tosatti-Weinkove [CTW17].

***Theorem 3.1* (Main Theorem). —**

(A) [Bou12]: *There is a constant $C > 0$ which does not depend on φ and A_0, A_1, A_2 such that*

$$\sup_X |\varphi| + \sup_{\partial X} |\nabla\varphi| \leq C. \tag{3.2}$$

(B) [Bou12]: *There exists a constant $C > 0$ only depending on A_2 such that*

$$\sup_X |\Delta\varphi| \leq C\left(1 + \sup_{\partial X} |\Delta\varphi|\right). \tag{3.3}$$

(C) [CKNS85, Gua98, Che00b, Bł12, Bou12]: *There exists $C(A_0, A_1) > 0$ such that*

$$\sup_{\partial X} |\nabla^2\varphi| \leq C\left(1 + \sup_X |\nabla\varphi|^2\right). \tag{3.4}$$

Furthermore, if we assume that the boundary of X is weakly pseudo-concave, then the constant C only depends on A_1.

(D) [Che00b, Bou12]: *There exists $C(A_0, A_1, A_2) > 0$ such that*

$$\sup_X |\nabla\varphi| \leq C. \tag{3.5}$$

If ∂X is weakly pseudo-concave, the constant C only depends on A_1 and A_2.

(E) [CTW17]: *There exists a constant $C(A_0, A_1, A_2) > 0$ such that*

$$\sup_X |\nabla^2\varphi| \leq C, \tag{3.6}$$

and $C = C(A_1, A_2)$ if the boundary is weakly pseudo-concave.

Smooth ω-psh solutions for the Dirichlet problem of Monge-Ampère operator are provided by the following subsolution criterion:

***Corollary 3.2* ([CKNS85, Gua98, Bou12]). —** *Let (X, ω) be an n-dimensional compact Kähler manifold with boundary. Given $f \in C^\infty(\partial X)$ and a smooth volume form μ, there exists a unique smooth ω-psh solution φ to the Dirichlet problem*

$$\begin{cases} (\omega + dd^c\varphi)^n = \mu \\ \varphi|_{\partial X} = f \end{cases} \tag{3.7}$$

if and only if there is a subsolution $\psi \in \mathrm{PSH}(X,\omega) \cap \mathcal{C}^\infty(\bar{X})$ *such that*

$$\begin{cases} (\omega + dd^c\psi)^n \geq \mu \\ \psi|_{\partial X} = f. \end{cases} \tag{3.8}$$

The main theorem also yields the $\mathcal{C}^{1,1}$-estimate for the geodesic problem. We say that a bounded function Φ on $X \times A$ is a weak geodesic connecting φ_0 and φ_1 if $\pi^*\omega + dd^c_{X\times A}\Phi \geq 0$ weakly on $X \times A$, and Φ solves (2.6) in the sense of Bedford-Taylor, i.e. in the sense of Radon measure.

Corollary 3.3 **([CTW17]).** — *Given any compact Kähler manifold* (X,ω) *and any two potentials in* $\mathcal{K}_\omega$*, the weak geodesic* Φ *connecting them belongs to* $\mathcal{C}^{1,1}(X \times A)$.

3.2. Strategy of the proof. — Now, we briefly explain the procedures and the ideas of proving main theorem.

(a) $\mathcal{C}^0$-estimate and boundary $\mathcal{C}^1$-estimate (A): The lower bound of φ is proved by the maximum principle for the complex Monge-Ampère operator. Then, we will construct an obstacle function h which only depends on background data to control the upper bound. Hence, the normal direction on the boundary is bounded by the normal derivative of h, and the tangential derivatives are controlled by assumption.

(b) Laplacian-estimate (B): This estimate is a direct consequence of Yau's celebrated inequality which we will state later (refer to page 350 in [Yau78]).

(c) Boundary $\mathcal{C}^2$-estimate (C): This part can be assort into three steps. First of all, the tangent-tangent derivative is obvious by assumption. Secondly, we will construct a barrier function w involving tangent-normal direction and then use maximum principle to control w. Finally, we introduce a lemma in [Bou12] which yields the normal-normal derivative immediately.

(d) $\mathcal{C}^1$-estimate (D): Concatenating (B) and (C), we obtain

$$\sup_X |\Delta\varphi| \leq C\left(1 + \sup_X |\nabla\varphi|^2\right). \tag{3.9}$$

Then, we will use a blow up argument to prove by contradiction.

(e) $\mathcal{C}^2$-estimate (E): This is the most complicated step. We apply the maximum principle to the quantity

$$Q = \log \lambda_1(\nabla^2\varphi) + h(|\partial\varphi|^2_\omega) - A\varphi$$

where $\lambda_1(\nabla^2\varphi)$ is the first eigenvalue of the matrix $\nabla^2\varphi$ and $h(s) = -\frac{1}{2}(1 + \sup_X |\partial\varphi|^2_\omega - s)$. Although the first eigenvalue may not be smooth, locally we can perturb $\nabla^2\varphi$ into another matrix with same smooth first eigenvalue around a maximum point of Q. Since the computation is quite complicated, we only sketch it in Section 6.

3.3. Toolbox. — In this subsection, we introduce some well-known tricks and they also play important roles in the proof of main estimates.

The maximum principle for the complex Monge-Ampère operator is a common technique to prove the uniqueness and it also plays an important role in the prove of $\mathcal{C}^0$-estimate (A).

***Lemma* 3.4 (Maximum Principle).** — *Let (X, ω) be a compact Kähler manifold with boundary and let φ_0, φ_1 are two smooth and strictly ω-psh functions such that*

$$\begin{cases} \varphi_0 \le \varphi_1 \text{ on } \partial X; \\ (\omega + dd^c\varphi_1)^n \le (\omega + dd^c\varphi_0)^n \text{ on } X. \end{cases} \tag{3.10}$$

Then, we have $\varphi_0 \le \varphi_1$ on X.

Proof of the maximum principle. — Write

$$0 \le (\omega + dd^c\varphi_0)^n - (\omega + dd^c\varphi_1)^n = dd^c(\varphi_0 - \varphi_1) \wedge T = L(\varphi_0 - \varphi_1)$$

where $T = \sum_{j=1}^{n-1}(\omega + dd^c\varphi_0)^j \wedge (\omega + dd^c\varphi_1)^{n-j-1}$ is a strictly positive smooth $(n-1, n-1)$-form and this implies that $L = dd^c(\bullet) \wedge T$ is a linear elliptic operator. Hence, $\varphi_0 - \varphi_1 \le 0$ by the linear elliptic maximum principle. □

The following inequality is a key ingredient for the second order estimate in Yau's proof of Calabi conjecture and we will use it to prove the Laplacian estimate (B).

***Lemma* 3.5 ([Yau78, p.350]).** — *If φ is a solution of equation (3.1), then φ has the following estimate*

$$\begin{aligned} &e^{B\varphi}\Delta_{\omega_\varphi} e^{-B\varphi}(n + \Delta\varphi) \\ \ge &\Delta F - n^2(\inf_{i\neq j} R_{i\bar{i}j\bar{j}}) - Bn(n+\Delta\varphi) + (B + \inf_{i\neq j} R_{i\bar{i}j\bar{j}})e^{-\frac{F}{n-1}}(n+\Delta\varphi)^{1+\frac{1}{n-1}} \end{aligned} \tag{3.11}$$

in the normal coordinate system, where B is a constant such that $B + (\inf_{i\neq j} R_{i\bar{i}j\bar{j}}) > 1$.

Implicit function theorem of Banach spaces and elliptic estimates are crucial tools in the proof of Corollary 3.2. One can see textbooks [**Aub98**] and [**GT01**] for more details.

***Lemma* 3.6 (Implicit Function Theorem for Banach Spaces)**
Let $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ be Banach spaces. Suppose $\mathfrak{F} : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ is $\mathcal{C}^k$ and let $D_y\mathfrak{F}(x_0, y_0) \in$ Hom$(\mathcal{Y}, \mathcal{Z})$ be the differential at y_0 of the mapping $y \mapsto \mathfrak{F}(x_0, y)$. If at $(x_0, y_0) \in \mathcal{X} \times \mathcal{Y}$, the linear map $D_y\mathfrak{F}(x_0, y_0)$ is invertible, then the map $(x, y) \mapsto (x, \mathfrak{F}(x, y))$ is a $\mathcal{C}^k$ diffeomorphism of a neighborhood $U_{(x_0,y_0)} \subset \mathcal{X} \times \mathcal{Y}$ of (x_0, y_0) onto an open set $\mathcal{X} \times \mathcal{Z}$.

***Lemma* 3.7 (Schauder Estimates).** — *Let Ω be a bounded $\mathcal{C}^{k+2,\alpha}$ domain, and suppose that $0 < \alpha < 1$ and $k \in \mathbb{N}$. For a uniformly elliptic second order operator L, there is a constant C depending only on L, k, α and the domains Ω such that if $L(u) = f$ and $u = v$ on $\partial\Omega$ for some $f \in \mathcal{C}^{k,\alpha}(\bar{\Omega})$, $v \in \mathcal{C}^{k+2,\alpha}(\bar{\Omega})$, then we have*

$$\|u\|_{\mathcal{C}^{k+2,\alpha}(\Omega)} \le C\left(\|u\|_{\mathcal{C}^0(\Omega)} + \|v\|_{\mathcal{C}^{k+2,\alpha}(\Omega)} + \|f\|_{\mathcal{C}^{k,\alpha}(\Omega)}\right).$$

3.4. $\mathcal{C}^0$-estimate and boundary $\mathcal{C}^1$-estimate (A). — We establish here the proof of (A). Recall that we assume $F \leq 0$. Immediately, we have $\varphi \geq 0$ by the maximum principle for the Monge-Ampère operator.

We now defined the obstacle function $h \in \mathcal{C}^\infty(\bar{X})$ as the solution to the Dirichlet problem

$$\begin{cases} \Delta h = -n \\ h|_{\partial X} = 0 \end{cases}. \tag{3.12}$$

Thus, we have

$$\Delta(\varphi - h) = \Delta\varphi + n = \mathrm{tr}_\omega(\omega + dd^c\varphi) \geq 0.$$

Also this implies $\varphi \leq h$ by the standard maximum principle. Therefore, we obtain the $\mathcal{C}^0$-estimate $0 \leq \varphi \leq h$. Regarding the boundary $\mathcal{C}^1$-estimate, since $\varphi|_{\partial X} = h|_{\partial X} = 0$ and $0 \leq \varphi \leq h$, the normal derivative of φ is bounded by the normal derivative of h and the tangential derivative is trivial.

3.5. Laplacian estimate (B). — We prove here (B). Let x_0 be a point where $e^{-B\varphi}(n + \Delta\varphi)$ achieves its maximum. If $x_0 \in \partial X$, then we have

$$\sup_X (n + \Delta\varphi) \leq e^{B(\sup_X \varphi - \inf_X \varphi)} \sup_{\partial X} (n + \Delta\varphi).$$

The oscillation of φ is bounded by $\mathcal{C}^0$-estimate (A). On the other hand, if x_0 in a interior maximum point, we then get

$$(n + \Delta\varphi)^{1+\frac{1}{n-1}} \leq e^{\frac{F}{n-1}} Bn(n + \Delta\varphi) + e^{\frac{F}{n-1}} \left(-\Delta F + n^2 (\inf_{i \neq j} R_{i\bar{i}j\bar{j}}) \right)$$

by the inequality of Yau in Lemma 3.5. Note that $y^{1+\frac{1}{n-1}} \leq ay + b$ implies either $y^{1+\frac{1}{n-1}} \leq 2ay$ or $y^{1+\frac{1}{n-1}} \leq 2b$. Therefore, we have $0 < (n + \Delta\varphi) \leq C(\sup F \leq 0, \sup(-\Delta F) = A_2)$ at x_0. Combing with the $\mathcal{C}^0$-estimate (A), we then complete the proof of the interior Laplacian estimate (B).

3.6. Apply to the existence criterion and the geodesic problem. —

Proof of Corollary 3.2. — After replacing ω by $\omega + dd^c\psi$ and φ by $\varphi - \psi$, we may assume $\psi = 0$, and $(\omega + dd^c\varphi)^n = e^F\omega^n$ for some smooth non-positive function F such that $\mu = e^F\omega^n$. Then, we follow the continuity method to complete the proof. The method of continuity proceeds by creating a one parameter family of equations which connect the equation of interest to an equation which solution is easily obtained. In our case, we choose the family of equations

$$(\omega + dd^c\varphi_t)^n = e^{tF}\omega^n, \ \ \omega + dd^c\varphi_t > 0, \text{ and } \ \varphi_t|_{\partial X} = 0 \tag{3.13}$$

parameterized by $t \in [0, 1]$. Denote by I the set of $t \in [0, 1]$ for which equation has a $\mathcal{C}^\infty$ solution. We have to prove three points:

(i) I is non-empty;
(ii) I is open in $[0, 1]$;
(iii) I is closed in $[0, 1]$.

Obviously, when $t = 0$, $\varphi_0 = 0$ is a solution, so $0 \in I$ and I is non-empty.

Step 1 Openness: To show the openness of I, we consider a solution φ_t where $t \in I$ and use the implicit function theorem for Banach space to say that there is an open neighborhood U_t which lies in I.

Now, we consider the non-linear map $\mathfrak{F}$ between two Banach spaces

$$\mathfrak{F} : [0,1] \times \left(\mathcal{C}^{2,\alpha}(X) \cap \{ u|_{\partial X} = 0\}\right) \longrightarrow \mathcal{C}^{0,\alpha}(X)$$

where

$$\mathfrak{F}(t,\varphi_t) := \log\left(\frac{(\omega + \mathrm{dd}^c\varphi_t)^n}{\omega^n}\right) - tF.$$

Hence, $\mathfrak{F}(0)$ is the set of $\mathcal{C}^{2,\alpha}$ solution to the Monge-Ampère equation (3.13). To apply the implicit function theorem, we must show the linearized operator of $\mathfrak{F}$ is invertible

$$D_2\mathfrak{F}_{(t,\varphi_t)}(v) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathfrak{F}(t,\varphi_t + \varepsilon v)\bigg|_{\varepsilon=0} = \Delta_{\omega_{\varphi_t}} v$$

where $v \in \mathcal{C}^{2,\alpha}(X) \cap \{ u|_{\partial X} = 0\}$. Since we only consider the zero boundary condition, $\ker\left(D_2\mathfrak{F}_{(t,\varphi_t)}\right) = 0$ by the maximum principle. On the other hand, surjectivity follows from standard elliptic theory. Hence, $D_2\mathfrak{F}_{(t,\varphi_t)}$ is invertible. By implicit function theorem, the map $(t,\psi) \mapsto (t,\mathfrak{F}(t,\psi))$ is a $\mathcal{C}^1$-diffeomorphism near (t,φ_t). Thus, for $\mathfrak{F} = 0$, we obtain a $\mathcal{C}^1$ map $(t-\varepsilon, t+\varepsilon) \to \mathcal{C}^{2,\alpha}(X) \cap \{ u|_{\partial X} = 0\}$ given by $t \mapsto \varphi_t$ with $\mathfrak{F}(t,\varphi_t) = 0$.

Next, we must prove that φ_t is smooth. We only have to say that $\partial_k\varphi_t$ satisfies a linear elliptic equation and then use elliptic regularity to conclude. Since $\mathfrak{F}(t,\varphi_t) = 0$, we have $\log\left(\frac{(\omega+\mathrm{dd}^c\varphi_t)}{\omega^n}\right) = tF$. Since $\varphi_t \in \mathcal{C}^{2,\alpha}$, we can use ∂_k to differentiate on both side, and then we obtain

$$\Delta_{\omega_{\varphi_t}}(\partial_k\varphi_t) = \partial_k(tF) + g^{i\bar{j}}\partial_k(g_{i\bar{j}}) - g^{i\bar{j}}_{\varphi_t}\partial_k(g_{i\bar{j}}).$$

Right hand side belongs to $\mathcal{C}^{0,\alpha}$ and the coefficients of $\Delta_{\omega_{\varphi_t}}$ are in $\mathcal{C}^{0,\alpha}$. Applying Schauder estimate, $\partial_k\varphi_t$ lies in $\mathcal{C}^{2,\alpha}$ for all k. Similarly, $\partial_{\bar{k}}\varphi_t \in \mathcal{C}^{2,\alpha}$ and hence $\varphi_t \in \mathcal{C}^{3,\alpha}$. Therefore, we have φ_t is smooth by the bootstrapping argument.

Step 2 Closedness: From Theorem 3.1, we already have the uniform $\mathcal{C}^2$-estimate. Evans-Krylov estimate yields the interior $\mathcal{C}^{2,\alpha}$ estimate for some $\alpha > 0$ as long as a $\mathcal{C}^2$ estimate is available. Similarly, Theorem 1 in [CKNS85] said that we have $\mathcal{C}^{2,\alpha}$ estimate up to the boundary in a similar situation.

To show I is closed, it is equivalent to say that if we can solve the equation (3.13) for all $t < t_0$, then we can take a limit and it also solve (3.13). Applying Arzela-Ascoli theorem, there is a subsequence $t_i \to t_0$ such that φ_{t_i} converges to φ_{t_0} in $\mathcal{C}^{2,\alpha'}$ for fixed $\alpha' < \alpha$. Note that φ_{t_0} satisfies

$$(\omega + \mathrm{dd}^c\varphi_{t_0})^n = e^{t_0F}\omega^n.$$

Similar to the bootstrapping argument which we already showed in the proof of openness, we can conclude that φ_{t_0} is smooth and hence $t_0 \in I$. These complete the proof of Corollary 3.2. □

Proof of Corollary 3.3. — Recall the degenerate complex Monge-Ampère related to the Dirichlet problem of geodesic equation (2.6): aims to find the function solving the following equation

$$\begin{cases} (\pi^*\omega + dd^c_{X\times A}\Phi)^{n+1} = 0, \\ \Phi(x,0) = \varphi_0(x), \\ \Phi(x,1) = \varphi_1(x). \end{cases}$$

The strategy is to find solutions Φ_ε of the "ε-geodesic" equations:

$$\begin{cases} \left(\pi^*\omega + i d\tau \wedge d\bar{\tau} + dd^c_{X\times A}(\Phi_\varepsilon - |\tau|^2)\right)^{n+1} = \varepsilon \pi^*\omega^n \wedge i d\tau \wedge d\bar{\tau}, \\ \Phi_\varepsilon(x,0) = \varphi_0(x), \\ \Phi_\varepsilon(x,1) = \varphi_1(x), \end{cases} \tag{3.14}$$

where τ is the parameter of the annulus $A := \{\tau \in \mathbb{C} \,|\, 1 < |\tau| < e\}$ and then let ε tend to zero to obtain the solution.

To solve ε-geodesic equations, we need to construct a subsolution to the Dirichlet problem (3.14) and then use Corollary 3.2 to obtain smooth solutions. Consider a smooth function $\chi : [0,1] \to [0,1]$ such that

$$\begin{cases} \chi(t) = 1 \text{ when } t \in \left[0, \frac{1}{4}\right], \\ \chi(t) = 0 \text{ when } t \in \left[\frac{3}{4}, 1\right]. \end{cases}$$

We also denote $\chi(\log|\tau|)$ by χ. Let u be a strictly subharmonic function on A with $u|_{\partial A} = 0$. For instance, we can choose

$$u(x) = -\log \mathrm{dist}(x, \partial A_\delta) - C_\delta$$

where $A_\delta = \{\tau \in \mathbb{C} \,|\, 1 - \delta < |\tau| < e + \delta\}$ and C_δ is a constant such that $u = 0$ on ∂A. Then, we have

$$\left(\pi^*\omega + i d\tau \wedge d\bar{\tau} + dd^c[\chi\varphi_0 + (1-\chi)\varphi_1 + Cu - |\tau|^2]\right)^{n+1} \geq \varepsilon(\pi^*\omega + i d\tau \wedge d\bar{\tau})^{n+1}$$

for C large enough and ε sufficiently small. To see this, it is sufficient to show $\pi^*\omega + i d\tau \wedge d\bar{\tau} + dd^c[\chi\varphi_0 + (1-\chi)\varphi_1 + Cu - |\tau|^2]$ is positive definite for C large enough. Indeed,

$$\begin{aligned} &\pi^*\omega + i d\tau \wedge d\bar{\tau} + dd^c[\chi\varphi_0 + (1-\chi)\varphi_1 + Cu - |\tau|^2] \\ &= \chi\pi^*\omega_{\varphi_0} + (1-\chi)\pi^*\omega_{\varphi_1} + i\left(\partial\chi \wedge \bar{\partial}(\varphi_0 - \varphi_1) + \partial(\varphi_0 - \varphi_1) \wedge \bar{\partial}\chi\right) + C dd^c u. \end{aligned}$$

Thus, it is reduced to a linear algebra exercise: can we enlarge C such that

$$\tilde{A} = \begin{bmatrix} A_{n\times n} & b_{n\times 1} \\ b^*_{1\times n} & C_{1\times 1} \end{bmatrix}_{(n+1)\times(n+1)}$$

is positive definite if we assume A to be positive definite? The answer is yes, because we can consider an elementary matrix $E = \begin{bmatrix} \mathrm{Id}_{n\times n} & 0 \\ -b^*A^{-1} & 1 \end{bmatrix}$ and then

$$E\tilde{A}E^* = \begin{bmatrix} A & 0 \\ 0 & C - b^*A^{-1}b \end{bmatrix}$$

is positive definite for C sufficiently large. Hence, $\chi\varphi_0 + (1-\chi)\varphi_1 + Cu - |\tau|^2$ is a subsolution and the boundary condition coincide with the boundary constraint of $\Phi_\varepsilon - |\tau|^2$, as desired.

Now, we have obtained solutions of the ε-geodesic equations for ε small enough by Corollary 3.2. From Theorem 3.1, we have the uniform estimate $\|\Phi_\varepsilon\|_{\mathcal{C}^2(X\times A)} \leq C$ for some uniform constant C which does not depends on the lower bound of ε. Hence, by Arzela-Ascoli theorem, there is a subsequence Φ_{ε_i} converging to Φ in $\mathcal{C}^1$. Since $\nabla\Phi_\varepsilon$ is 1-Lipschitz with uniform Lipschitz constant, we can conclude that Φ lies in $\mathcal{C}^{1,1}(X \times A)$ and it is a solution to Dirichlet problem of degenerate complex Monge-Ampère equation (2.6) by the continuity properties of complex Monge-Ampère operator. □

4. Proof of Main Theorem: boundary $\mathcal{C}^2$-estimate

To prove the boundary $\mathcal{C}^2$-estimate (C), we have to consider three different directions: tangent-tangent, tangent-normal, and normal-normal derivatives. Note that locally the tangential operators are

$$D_k = \frac{\partial}{\partial x_k} - \frac{r_{x_k}}{r_{x_{2n}}}\frac{\partial}{\partial x_{2n}}$$

for $k = 1, \cdots, 2n-1$ and the normal operator is

$$D_{2n} = -\frac{1}{r_{x_{2n}}}\frac{\partial}{\partial x_{2n}}.$$

Indeed, these form a dual basis corresponding to $\mathrm{d}x^1, \cdots, \mathrm{d}x^{2n-1}, \mathrm{d}r$.

4.1. Tangent-tangent derivative $D_iD_j\varphi$. — This part is trivial by the boundary assumption $\varphi|_{\partial X} = 0$.

4.2. Tangent-normal derivative $D_{2n}D_k\varphi$. — To prove the tangent-normal direction, we first consider a barrier function of the form

$$v := \varphi + sh - Nd^2,$$

in a boundary local chart $\Omega_\delta = X \cap B_\delta(0)$ and 0 corresponds to the point which we want to estimate on the boundary, where d is the distance function to the boundary. The main ingredient is the following lemma.

***Lemma 4.1* ([Gua98, Che00b]). —** *For N sufficiently large and s, δ sufficiently small,*

$$\Delta_{\omega_\varphi} v \leq -\frac{\varepsilon}{4}\left(1 + \sum_{\alpha=1}^{n} \tilde{g}^{\alpha\bar{\alpha}}\right)$$

in Ω_δ, and $v \geq 0$ on $\partial\Omega_\delta$, where ε is a local lower bound of ω, $\tilde{g}$ is the associated metric of ω_φ.

Proof of Lemma 4.1. — Note that $g_{j\bar{k}} \geq \varepsilon\delta_{jk}$ in Ω_δ. Then, we have

$$\Delta_{\omega_\varphi}\varphi = \sum_{j,k}\tilde{g}^{j\bar{k}}\left(g_{j\bar{k}} + \varphi_{j\bar{k}} - g_{j\bar{k}}\right) \leq n - \varepsilon\sum_{\alpha}\tilde{g}^{\alpha\bar{\alpha}}.$$

and

$$\Delta_{\omega_\varphi} h \le C_1\left(1+\sum_\alpha \tilde{g}^{\alpha\bar{\alpha}}\right)$$

for some constant C_1 which only depends on the background data. Moreover,

$$\Delta_{\omega_\varphi} Nd^2 = 2N(dN\Delta_{\omega_\varphi} d + \tilde{g}^{j\bar{k}} d_j d_{\bar{k}}).$$

Observe that

$$\Delta_{\omega_\varphi} d \ge -C_2\left(1+\sum_\alpha \tilde{g}^{\alpha\bar{\alpha}}\right)$$

for some constant C_2 depending only on the background data. On the other hand, since $d_{x_k}(0)=0$ for all $k<2n$ and $d_{x_{2n}}(0)=1$, we have, for δ small enough

$$\sum_{j,k}\tilde{g}^{j\bar{k}} d_j d_{\bar{k}} \ge \tilde{g}^{n\bar{n}} d_n d_{\bar{n}} + \sum_{k<n}(\tilde{g}^{n\bar{k}} d_n d_{\bar{k}} + \tilde{g}^{k\bar{n}} d_k d_{\bar{n}}) \ge \frac{\tilde{g}^{n\bar{n}}}{8} - C_3\delta\sum_\alpha \tilde{g}^{\alpha\bar{\alpha}}$$

in Ω_δ after shrinking δ. Suppose $0<\lambda_1\le\lambda_2\le\cdots\le\lambda_n$ are eigenvalues of $\tilde{g}_{j\bar{k}}$. Then, $\sum_\alpha \tilde{g}^{\alpha\bar{\alpha}} = \sum_\alpha \lambda_\alpha^{-1}$ and $\tilde{g}^{n\bar{n}} \ge \lambda_n^{-1}$. Hence,

$$\begin{aligned}&\Delta_{\omega_\varphi} v\\ \le& n - \varepsilon\sum_\alpha \tilde{g}^{\alpha\bar{\alpha}} + sC_1\left(1+\sum_\alpha \tilde{g}^{\alpha\bar{\alpha}}\right) + 2N\left(\delta(C_2+C_3)\left(1+\sum_\alpha \tilde{g}^{\alpha\bar{\alpha}}\right) - \frac{\tilde{g}^{n\bar{n}}}{8}\right).\end{aligned} \tag{4.1}$$

Note that

$$\frac{\varepsilon}{4}\sum_\alpha \tilde{g}^{\alpha\bar{\alpha}} + \frac{N}{4}\tilde{g}^{n\bar{n}} \ge \frac{n\varepsilon}{4}N^{\frac{1}{n}}\left(\lambda_1^{-1}\cdots\lambda_n^{-1}\right)^{\frac{1}{n}} \ge C_4 N^{\frac{1}{n}}$$

for ε small enough. Now we fix $s>0$ sufficiently small such that $sC_1\le\frac{\varepsilon}{4}$, and choose N sufficiently large so that $-C_4N^{\frac{1}{n}} + n + sC_1 \le -\frac{\varepsilon}{4}$. After shrinking δ again such that $2N\delta(C_2+C_3)\le\frac{\varepsilon}{4}$, we obtain

$$\Delta_{\omega_\varphi} v \le -\frac{\varepsilon}{4}\left(1+\sum_\alpha \tilde{g}^{\alpha\bar{\alpha}}\right)$$

in Ω_δ.

Recall that $\Delta h = -n$. Then there exists a constant c which depends only on ω such that $h > cd$ in Ω_δ. On $X\cap\partial B_\delta(0)$, $v \ge scd - Nd^2 \ge (sc-N\delta)d \ge 0$ for δ sufficiently small. On $\partial X\cap B_\delta(0)$, we have $v\equiv 0$. These complete the proof of the key lemma. □

The barrier function w which we mainly consider to get the control on $D_{2n}D_k\varphi$ is

$$w := D_k\varphi - C'\varphi_{x_{2n-1}}^2 + Av + B|z|^2$$

where A, B and C' are constants to be determined and $D_k := \frac{\partial}{\partial x_k} + a\frac{\partial}{\partial x_{2n}}$.

We claim that

$$w \ge 0 \text{ on } \Omega_\delta \text{ and } w(0)=0.$$

If this is the case, we can control the tangent-normal derivatives as follows. Clearly,

$$\frac{\partial}{\partial x_{2n}} w(0) \le 0$$

and this implies

$$\frac{\partial}{\partial x_{2n}} D_k\varphi(0) \le -A\frac{\partial}{\partial x_{2n}} v(0)$$

because we can eliminate the effect of $\varphi_{x_{2n-1}}(0)$ after some rotation of coordinates such that $D_{2n-1} = \frac{\partial}{\partial x_{2n-1}}$ at 0. We also claim that $A \le CM$ where $M = (1 + \sup_X |\nabla\varphi|^2)$ and C is a constant which only depends on background data. Recall that $v = \varphi + sh - Nd^2$, so everything is controlled by boundary $\mathcal{C}^1$-estimate (A) and given data.

Now, we work out the detail of the claim. Note that

$$\begin{aligned}\Delta_{\omega_\varphi}(D_k\varphi) &= \mathrm{tr}_{\omega_\varphi}(D_k dd^c\varphi) + (\Delta_{\omega_\varphi} a)\varphi_{x_{2n}} + \mathrm{tr}_{\omega_\varphi}(\partial a \wedge \bar\partial\varphi_{x_{2n}} + \partial\varphi_{x_{2n}} \wedge \bar\partial a)\\ &= D_k F + \mathrm{tr}_{\omega}(D_k\omega) - \mathrm{tr}_{\omega_\varphi}(D_k\omega) + (\Delta_{\omega_\varphi} a)\varphi_{x_{2n}} + \mathrm{tr}_{\omega_\varphi}(\partial a \wedge \bar\partial\varphi_{x_{2n}} + \partial\varphi_{x_{2n}} \wedge \bar\partial a)\end{aligned}$$

Observe that $\tilde g^{p\bar q}\tilde g_{n\bar q} = \delta^p_n$ and $\frac{\partial}{\partial x_{2n}} = 2\mathrm{i}\frac{\partial}{\partial z_n} - \mathrm{i}\frac{\partial}{\partial x_{2n-1}}$ and we set $t = x_{2n-1}$. Then $\varphi_{x_{2n}\bar q} = 2\mathrm{i}\varphi_{n\bar q} - \mathrm{i}\varphi_{x_{2n-1}\bar q}$. Thus, by Cauchy-Schwarz, we have

$$\begin{aligned}\left|\mathrm{tr}_{\omega_\varphi}(\partial a \wedge \bar\partial\varphi_{x_{2n}})\right| &= \left|\tilde g^{j\bar k} a_j(\varphi_{x_{2n}\bar k})\right| = \left|\tilde g^{j\bar k} a_j(2\varphi_{n\bar k} - \varphi_{t\bar k})\right|\\ &= \left|2a_n - 2\tilde g^{j\bar k} a_j g_{n\bar k} - \tilde g^{j\bar k} a_j\varphi_{t\bar k}\right|\\ &\le C_5\left[1 + \left(\sum_\alpha \tilde g^{\alpha\bar\alpha}\right)^{\frac12}\left(\tilde g^{j\bar k}(1 + \varphi_{jt}\varphi_{\bar k t})\right)^{\frac12}\right]\\ &\le C_5\left[1 + \sum_\alpha \tilde g^{\alpha\bar\alpha} + \tilde g^{j\bar k}\varphi_{jt}\varphi_{\bar k t}\right]\end{aligned} \tag{4.2}$$

where C_5 is a constant only depending on $\sup_{\Omega_\delta}|\nabla a|$. Combining all of these ingredients, we obtain

$$\Delta_{\omega_\varphi}(D_k\varphi) \le C\left(1 + \sum_\alpha \tilde g^{\alpha\bar\alpha}\right) + 2C_5\left[1 + \sum_\alpha \tilde g^{\alpha\bar\alpha} + \tilde g^{j\bar k}\varphi_{jt}\varphi_{\bar k t}\right] + C_6 M\left(\sum_\alpha \tilde g^{\alpha\bar\alpha}\right) \tag{4.3}$$

where $C = C(A_1)$ and C_6 is a constant only depending on $\sup_{\Omega_\delta}|\nabla^2 a|$. Furthermore,

$$\begin{aligned}\Delta_{\omega_\varphi}\varphi_t^2 &= 2\tilde g^{j\bar k}\varphi_{jt}\varphi_{\bar k t} + 2\varphi_t\left(F_t + g^{j\bar k}\partial_t g_{j\bar k} - \tilde g^{j\bar k}\partial_t g_{j\bar k}\right)\\ &\ge 2\tilde g^{j\bar k}\varphi_{jt}\varphi_{\bar k t} - 2CM\left(1 + \sum_\alpha \tilde g^{\alpha\bar\alpha}\right)\end{aligned} \tag{4.4}$$

Then, we choose $C' = C_5$ to eliminate $\tilde g^{j\bar k}\varphi_{jt}\varphi_{\bar k t}$, so we have

$$\Delta_{\omega_\varphi} w \le \left(-\frac{\varepsilon A}{4} + B + C + 2C_5 + C_6 M + 2CC_5 M\right)\left(1 + \sum_\alpha \tilde g^{\alpha\bar\alpha}\right) \tag{4.5}$$

by previous steps and Lemma 4.1.

Now, we can fix B large enough such that $w \ge 0$ on $\partial\Omega_\delta$, since $v \ge 0$ is proved. Note that this B depends on M of order 1. Then, we choose A sufficiently large to get $\Delta_{\omega_\varphi} w \le 0$ and A depends on M of order 1. In sum, $w \ge 0$ on Ω_δ by maximum principle and $w(0) = 0$. This completes the proof of the claim. Finally, we replace D_k by $-D_k$ to get the other side bound.

4.3. Normal-normal derivative $D_{2n}D_{2n}\varphi$. — Before proving the uniform bound on Normal-Normal derivative, we have to introduce a key lemma.

Lemma 4.2 **([CKNS85, Gua98, Bou12]). —** *There exists $\varepsilon > 0$ only depending on A_0 such that*

$$(\omega + dd^c\varphi)|_{T^h_{\partial X}} \geq \varepsilon\, \omega|_{T^h_{\partial X}}.$$

In particular, if ∂X is weakly pseudo-concave, we can even take $\varepsilon = 1$.

We first use Lemma 4.2 to show the normal-normal estimate and then back to prove the lemma in next subsection.

Since we already showed that $D_iD_j\varphi(0)$ and $D_{2n}D_j\varphi(0)$ in Tangent-Tangent estimate, and Tangent-Normal estimate, to show $|D_{x_{2n}}D_{x_{2n}}\varphi(0)| \leq CM$ is equivalent to prove that $|\varphi_{n\bar{n}}(0)| \leq CM$, where $M = (1 + \sup_X |\nabla\varphi|^2)$ again. Note that we have

$$|\varphi_{k\bar{n}}(0)| \leq CM$$

for all $k = 1, \cdots, n-1$. Expanding out the determinant $\det(\tilde{g}_{j\bar{k}})_{1\leq j,k\leq n}$ at 0 by linear algebra, we have

$$\det(\tilde{g}_{j\bar{k}})_{1\leq j,k\leq n} = \tilde{g}_{n\bar{n}} \det(\tilde{g}_{j\bar{k}})_{1\leq j,k\leq n-1} + \sum_j (-1)^{n-j}\tilde{g}_{n\bar{j}} \det(A_j)$$

where A_j is controlled by CM since it only involves tangent-tangent and tangent-normal derivatives. Then, we obtain

$$\left|\det(\tilde{g}_{j\bar{k}})_{1\leq j,k\leq n} - \tilde{g}_{n\bar{n}} \det(\tilde{g}_{j\bar{k}})_{1\leq j,k\leq n-1}\right| \leq CM \tag{4.6}$$

at 0. Recall that the Monge-Ampère equation is $(\omega + dd^c\varphi)^n = e^F\omega^n$ and $F \leq 0$ by assumption. We have

$$0 \leq \det(\tilde{g}_{j\bar{k}})_{1\leq j,k\leq n} = \det(g_{j\bar{k}} + \varphi_{j\bar{k}})_{1\leq j,k\leq n} \leq \det(g_{j\bar{k}})_{1\leq j,k\leq n} \leq C.$$

These imply

$$\left|\tilde{g}_{n\bar{n}} \det(\tilde{g}_{j\bar{k}})_{1\leq j,k\leq n-1}\right| \leq CM. \tag{4.7}$$

Next, we show that there is a uniform lower bound for $\det(\tilde{g}_{j\bar{k}})_{1\leq j,k\leq n-1}$. Recall that $T^h_{\partial X}$ is spanned by $\{\frac{\partial}{\partial z^1}, \cdots \frac{\partial}{\partial z^{n-1}}\}$ at 0 and by Lemma 4.2

$$(\omega + dd^c\varphi)|_{T^h_{\partial X}} \geq \varepsilon(A_0)\, \omega|_{T^h_{\partial X}}.$$

Obviously, we get

$$\det(\tilde{g}_{j\bar{k}})_{1\leq j,k\leq n-1} \geq C''(A_0) := \varepsilon^{n-1}(A_0) \tag{4.8}$$

at 0. In sum, combining (4.7) and (4.8), we have

$$|\varphi_{n\bar{n}}| \leq |\tilde{g}_{n\bar{n}} - g_{n\bar{n}}| \leq \frac{CM}{C''(A_0)} \tag{4.9}$$

at 0, as desired.

4.4. Proof of Lemma 4.2. —

Proof of Lemma 4.2. — First of all, we begin from the weakly pseudo-concave cases. We have $\varphi \geq 0$ from the $\mathcal{C}^0$-estimate (A) and $\varphi|_{\partial X} = 0$ by the assumption. Thus, $\nu \cdot \varphi < 0$ and $\nu \cdot r > 0$ on ∂X. Recall that

$$\mathrm{dd}^c \varphi|_{T^h_{\partial X}} = \frac{\nu \cdot \varphi}{\nu \cdot r}\, \mathrm{dd}^c r|_{T^h_{\partial X}} .$$

Since ∂X is weakly pseudo-concave, namely $\mathrm{dd}^c r|_{T^h_{\partial X}} \leq 0$, we obtain

$$\mathrm{dd}^c \varphi|_{T^h_{\partial X}} \geq 0$$

and hence

$$(\omega + \mathrm{dd}^c \varphi)|_{T^h_{\partial X}} \geq \omega|_{T^h_{\partial X}} .$$

Before proving the general cases, we put some simple remarks

Remark 4.1. — We can choose a coordinate on $B_R(0)$ centered at 0 such that

$$r = -x_{2n} + \Re\left(\sum_{1\leq j,k\leq n} a_{jk} z_j \bar{z}_k\right) + O(|z|^3). \tag{4.10}$$

One can see [**Bou12**, p.266] for more details.

Remark 4.2. — Assume that $r_{x_k}(0) = 0$ for all $k = 1, \cdots, 2n-1$ and $r_{x_{2n}}(0) = -1$. By simple calculus, we have

$$(f|_{\partial X})_{x_k|_{\partial X}}(0) = f_{x_k}(0) + r_{x_k}(0) f_{x_{2n}}(0) \tag{4.11}$$

and

$$(f|_{\partial X})_{x_j|_{\partial X} x_k|_{\partial X}}(0) = f_{x_j x_k}(0) + r_{x_j x_k}(0) f_{x_{2n}}(0) \tag{4.12}$$

for any given smooth function f near 0. In particular, if we consider a smooth function f on X and $f|_{\partial X} = 0$, then we have

$$\mathrm{dd}^c f|_{T^h_{\partial X}} = \frac{\nu \cdot f}{\nu \cdot r}\, \mathrm{dd}^c r|_{T^h_{\partial X}} \tag{4.13}$$

where ν is the outward pointing unit normal vector field to ∂X.

Now, we are going to prove the general case and say that ε is a constant only depending on A_0. It suffices to show that there exists $\varepsilon(A_0)$ such that

$$(\omega + \mathrm{dd}^c \varphi)|_{T^h_{\partial X}} \geq \varepsilon\, \mathrm{dd}^c r|_{T^h_{\partial X}} . \tag{4.14}$$

We prove this claim first. According to Remark 4.2, we have

$$(\mathrm{dd}^c \varphi)|_{T^h_{\partial X}} = \frac{\nu \cdot \varphi}{\nu \cdot r}\, \mathrm{dd}^c r|_{T^h_{\partial X}} .$$

By the boundary $\mathcal{C}^1$-estimate (A), there is a constant C only depending background data such that $-C \leq \frac{\nu\cdot\varphi}{\nu\cdot r} \leq 0$ on ∂X. Then,

$$(\omega + \mathrm{dd}^c \varphi)|_{T^h_{\partial X}} \geq \varepsilon\, \mathrm{dd}^c r|_{T^h_{\partial X}} \geq \frac{\varepsilon}{-C}\, \mathrm{dd}^c \varphi|_{T^h_{\partial X}} = \frac{\varepsilon}{-C}\, (\omega + \mathrm{dd}^c \varphi - \omega)|_{T^h_{\partial X}}$$

and this implies the desired result

$$(\omega + dd^c\varphi)|_{T^h_{\partial X}} \geq \frac{\varepsilon}{C}\left(1+\frac{\varepsilon}{C}\right)^{-1} \omega|_{T^h_{\partial X}}.$$

To prove $(\omega + dd^c\varphi)|_{T^h_{\partial X}} \geq \varepsilon\, dd^c r|_{T^h_{\partial X}}$ (4.14), it is enough to consider $dd^c r(v) > 0$ where $v = \sum_{j\leq n} v_j \frac{\partial}{\partial z_j} \in T^h_{\partial X,0}$ with $\sum |v_j|^2 = 1$. We may assume $v = \frac{\partial}{\partial z_1}$ and the coefficients in (4.10) $(a_{jk})_{1\leq j,k\leq n}$ is diagonal after some unitary transformation.

Step 1 Good choice of Kähler potential: Let ρ be a potential function with $dd^c\rho = \omega$ and $\rho(0) = 0$. We claim that ρ can be chosen to satisfy the following estimate

$$\rho|_{T^h_{\partial X}} \leq \Re\left(\sum_{j=2}^{n} c_j z_1 \bar{z}_j\right) + O(|z_2|^2 + \cdots + |z_n|^2). \tag{4.15}$$

Recall the elementary formula in Remark 4.2. We have

$$(f|_{\partial X})_{z_1|_{\partial X}\bar{z}_k|_{\partial X}}(0) = f_{z_1\bar{z}_k}(0) + \delta_{1k} a_{11} f_{x_{2n}}(0)$$

for all $k < n$. Applying on the formula

$$0 = r = x_{2n} + \Re\left(\sum_{1\leq j,k\leq n} a_{jk} z_j \bar{z}_k\right) + O(|z|^3),$$

we get

$$x_{2n}|_{\partial X} = a_{11}|z_1|^2 + x_{2n-1}\Re(bz_1) + q(x') + O(|z_1|^4 + |z_2|^2 + \cdots |z_{n-1}|^2 + x_{2n-1}^2) \tag{4.16}$$

where q is a homogeneous cubic polynomial and each term involving x_1 or x_2 of order 2 and $x' = (x_1, x_2, \cdots, x_{2n-1})$. Note that, in $q(x')$, the cubic term on (x_1, x_2) has a unique decomposition

$$\Re(a' z_1^3 + b' z_1 |z_1|^2),$$

the terms that are quadratic in (x_1, x_2) can be written in the form

$$\Re\left(\sum_{j=2}^{n-1} z_1^2 (a'_{1j} z_j + a'_{1\bar{j}} \bar{z}_j)\right) + \Re\left(\sum_{j=2}^{n-1} b'_j z_j |z_1|^2\right).$$

After shrinking the radius δ of the coordinate chart $\Omega_\delta := X \cap B_\delta(0)$, we have

$$(a_{11} - \varepsilon)|z_1|^2 \leq x_{2n} - x_{2n-1}\Re(bz_1) + O(|z_2|^2 + \cdots + |z_{n-1}|^2 + x_{2n-1}^2) \leq (a_{11}+\varepsilon)|z_1|^2, \tag{4.17}$$

because we assume $dd^c r(v) = a_{11} > 0$ in the beginning . Regarding the potential function ρ, we may assume that

$$(\rho|_{\partial X})_{z_1|_{\partial X}\bar{z}_1|_{\partial X}} = \rho_{z_1\bar{z}_1}(0) + \rho_{x_{2n}}(0) a_{11} = 0. \tag{4.18}$$

Indeed, we can replace ρ by $\rho - \lambda x_{2n}$ and choose appropriate λ such that $\rho_{z_1\bar{z}_1}(0) + \rho_{x_{2n}}(0)a_{11} = 0$. Note that this modification does not change the condition $dd^c\rho = \omega$. Hence, on $\partial X \cap B_\delta(0)$, ρ is expanded in a Taylor series

$$\rho|_{\partial X} = \sum_{1\leq\alpha,\beta\leq 2n-1} \gamma_{\alpha\beta} x_\alpha x_\beta + p(x') + O(x').$$

where p is a homogeneous cubic polynomial. We may assume $\gamma_{11} = \gamma_{12} = \gamma_{22} = 0$. Indeed,

$$\gamma_{11}x_1^2 + 2\gamma_{12}x_1x_2 + \gamma_{22}x_2^2 = \Re(\tilde{a}z_1^2) + \tilde{b}\,|z_1|^2$$

for some constant $\tilde{a} \in \mathbb{C}$, $\tilde{b} \in \mathbb{R}$ and we already have $b = 0$ from (4.18). Note that $dd^c\rho = \omega$ does not change after subtracting a real part of holomorphic polynomial. Therefore we can drop terms involving x_1^2, x_1x_2, x_2^2. To see the claim, we note that

$$\sum_{\alpha=1}^{2}\sum_{\beta=3}^{2n-1}\gamma_{\alpha\beta}x_\alpha x_\beta = \Re\left(\sum_{j=1}^{n-1} z_1(\tilde{a}_{1j}z_j + \tilde{a}_{1\bar{j}})\right) + \Re(\tilde{c}z_1x_{2n-1}).$$

Thus,

$$\sum_{1\le\alpha,\beta\le 2n-1}\gamma_{\alpha\beta}x_\alpha x_\beta = \Re\left(\sum_{j=2}^{n-1} z_1(\tilde{a}_{1j}z_j + \tilde{a}_{1\bar{j}}\bar{z}_j)\right) + \Re(\tilde{c}z_1x_{2n-1}) + O(x_3^2 + \cdots x_{2n-1}^2).$$

Next, in $p(x')$, the cubic in (x_1, x_2) has a unique decomposition $\Re(Az_1^3 + Bz_1\,|z_1|^2)$, the terms that are quadratic in (x_1, x_2) can be written in the form

$$\Re\left(\sum_{j=2}^{n-1} z_1^2(a''_{1j}z_j + a''_{1\bar{j}}\bar{z}_j)\right) + \Re\left(\sum_{j=2}^{n-1} c''_j z_j\,|z_1|^2\right)$$

and all the other terms are bounded by $C\sum_{3\le\beta\le 2n-1} x_\beta^2$. Finally, using (4.17), we can replace $|z_1|^2$ by $x_{2n} - x_{2n-1}\Re(bz_1) + O(x_3^2 + \cdots + x_{2n-1}^2)$, combine everything and subtract real part of holomorphic polynomials to get

$$\rho|_{T^h_{\partial X}} \le \Re\left(\sum_{j=2}^{n} c_j z_1\bar{z}_j\right) + O(|z_2|^2 + \cdots + |z_n|^2).$$

Step 2 Barrier construction: We are going to show the existence of a good barrier function b which satisfies $b \ge \rho + \varphi$ on Ω_δ We consider the barrier function b defined by

$$b(z_1, \cdots, z_n) := -\varepsilon_1 x_{2n} + \varepsilon_2\,|z|^2 + \frac{1}{\mu}\sum_{j=2}^{n}\left|c_j z_1 + \mu z_j\right|^2$$

where $\varepsilon_1, \varepsilon_2$ and μ are constants to be determined. Now, we want to show that we can shrink the radius of Ω_δ and then choose $\varepsilon_1, \varepsilon_2$ and μ only depending on A_0 such that

$$b \ge \rho + \varphi. \tag{4.19}$$

on Ω_δ. Recall that $r = -x_{2n} + \Re\left(\sum_{1\le j,k\le n} a_{jk}z_j\bar{z}_k\right) + O(|z|^3)$ and $\Omega_\delta \subset \{r < 0\}$, so we have

$$|z'| \ge |z_m| \ge x_m \ge \Re\left(\sum_{1\le j,k\le n} a_{jk}z_j\bar{z}_k\right) + O(|z|^3) \tag{4.20}$$

where $z' = (z_2, \cdots, z_n)$. We are going to show that $b \ge \rho + \varphi$ on $\partial\Omega_\delta$ first, and then use the maximum principle to conclude the desired inequality on whole Ω_δ.

On $\partial B_\delta(0) \cap X$, we have $|z| = \delta$ and $a_{11} > 0$ by assumption. We can shrink δ such that there is a constant $\beta > 0$ with

$$|z'|^2 \ge \beta \tag{4.21}$$

on $\partial B_\delta(0) \cap X$. Otherwise, theses is a point $p \in \partial B_\delta(0) \cap X$ such that $|z'|^2 = 0$ and this implies

$$0 \geq a_{11} |z_1|^2 + O(|z_1|^3) = a_{11}\delta^2 + O(\delta^3)$$

by (4.20) and this is not true for δ sufficiently small because we assume $a_{11} > 0$.

On $\partial X \cap B_\delta(0)$, note that $r = 0$. Therefore, we can find a constant $C > 0$ such that

$$-\varepsilon_1 x_{2n} + \varepsilon_2 |z|^2 \geq 0 \tag{4.22}$$

on $\partial X \cap B_\delta(0)$, since $-x_{2n} + \Re\left(\sum_{1\leq j,k\leq n} a_{jk} z_j \bar{z}_k\right) + O(|z|^3) = 0$ and this implies

$$-x_{2n} + C|z|^2 \geq 0$$

for some C sufficiently large. We pick $\varepsilon_2 = C\varepsilon_1$.

Now, we are going to show that we can find some suitable μ such that $b \geq \rho + \varphi$ on $\partial\Omega_\delta$. Note that

$$\frac{1}{2\mu}\sum_{j=2}^{n} |c_j z_1 + \mu z_j|^2 = \frac{1}{2\mu}\sum_{j=2}^{m} |c_j z_1|^2 + \Re\left(\sum_{j=2}^{n} c_j z_1 \bar{z}_j\right) + \frac{\mu}{2}|z'|^2$$

the second term on the right hand side is equal to the leading order term of ρ in the expansion (4.15), so we can choose μ large enough to control ρ. On $\partial X \cap B_\delta(0)$, $\varphi = 0$ by assumption and (4.22), $-\varepsilon_1 |z_1| + \varepsilon_2 |z| \geq 0$ on $\partial X \cap B_\delta(0)$, we then have $b \geq \rho + \varphi$ on $\partial X \cap B_\delta(0)$. On the other hand, we already have $\sup_X |\varphi| < C$ from the $\mathcal{C}^0$-estimate (A) and

$$b \geq -\varepsilon_1 x_{2n} + \varepsilon_2\delta^2 + \frac{\mu}{2}|z'|^2 \geq -\varepsilon_1 x_{2n} + \varepsilon_2\delta^2 + \frac{\mu}{2}\beta$$

on $\partial B_\delta(0) \cap X$ by (4.22). Then, we choose μ sufficiently large to dominate ρ and φ.

Next, we pick ε_1 sufficiently small and only depending on μ and A_0 such that

$$(\mathrm{dd}^c b)^n \leq e^{-A_0}\omega^n \leq e^F\omega^n = (\mathrm{dd}^c(\rho + \varphi))^n.$$

Indeed, since

$$\mathrm{dd}^c(b) = \varepsilon_2 \sum_{j=1}^{n} \mathrm{d}z^j \wedge \mathrm{d}\bar{z}^j + \mathrm{dd}^c\left(\frac{1}{\mu}\sum_{j=2}^{n} |c_j z_1 + \mu z_j|\right)$$

and

$$\left(\mathrm{dd}^c\left(\frac{1}{\mu}\sum_{j=2}^{n} |c_j z_1 + \mu z_j|\right)\right)^n = 0,$$

we have $(\mathrm{dd}^c b)^n = O(\varepsilon_2) = O(\varepsilon_1)$. Finally, we get $b \geq \rho + \varphi$ on Ω_δ by the maximum principle of complex Monge-Ampère operator.

Step 3 Conclusion: Because we already have $b \geq \rho + \varphi$ on Ω_δ and $b(0) = \rho(0) + \varphi(0)$, we then obtain

$$(\rho + \varphi)_{x_{2n}}(0) \leq b_{x_{2n}}(0) = -\varepsilon_1.$$

Using the assumption of ρ, (4.18), and the direct calculation of implicit function derivative (4.2), we have

$$\rho_{z_1\bar{z}_1}(0) + \rho_{x_{2n}}(0)a_{11} = 0$$

and similarly due to $\varphi|_{\partial X} = 0$ we get

$$\varphi_{z_1\bar{z}_1}(0) + \varphi_{x_{2n}}(0)a_{11} = 0.$$

In sum,

$$(\omega + dd^c\varphi)(v) = (\rho + \varphi)_{z_1\bar{z}_1}(0) \geq \varepsilon_1 a_{11} = \varepsilon_1 dd^c r(v).$$

This completes the proof of Lemma 4.2. □

5. Proof of Main Theorem: $\mathcal{C}^1$-estimate

By the interior Laplacian estimate (B) and the boundary second order estimate (C), we have

$$\sup_X |\Delta\varphi| \leq C\left(1 + \sup_X |\nabla\varphi|^2\right), \tag{5.1}$$

where C depends on A_0, A_1 and A_2 (depending only on A_1 and A_2 if ∂X is weakly pseudo-concave). Suppose the $\mathcal{C}^1$-estimate fails, i.e. there exists a sequence of functions φ_j and points $p_j \in X$ such that

$$\left|\nabla\varphi_j(p_j)\right| = \sup_X \left|\nabla\varphi_j\right| =: C_j \to +\infty.$$

We want to draw a contradiction from this statement. First, we may assume $p_j \to p$ by compactness. Then, choose a domain $\Omega_\delta := X \cap B_\delta(0)$ such that p corresponds to 0 in this local coordinate. We define

$$\tilde{\varphi}_j(z) = \varphi_j\left(p_j + \frac{1}{C_j}z\right)$$

for all $z \in \Omega_{\delta C_j} := X \cap B_{\delta C_j}(0)$. Thus we have

$$\left|\nabla\tilde{\varphi}_j(0)\right| = 1 \text{ and } \sup_{\Omega_{\delta C_j}} \left|\Delta\tilde{\varphi}_j\right| \leq 2C \text{ by (5.1)}.$$

Using standard elliptic estimate and Rellich embedding theorem, we have $(\tilde{\varphi}_j)_j$ is in a compact set of $\mathcal{C}^1(\Omega_\delta)$. Hence, there exists a subsequence of $\tilde{\varphi}_j$ a limit function $\tilde{\varphi}$ in $\mathbb{C}^{n+1}$ (or half ball if p is on ∂X) such that in any fixed domain $\Omega_R = X \cap B_R(0)$ we have $\tilde{\varphi}_j \to \tilde{\varphi}$ in $\mathcal{C}^1$. This implies

$$|\nabla\tilde{\varphi}(0)| = 1. \tag{5.2}$$

We can do the same rescale on h and define

$$\tilde{h}_j(z) = h\left(p_j + \frac{1}{C_j}z\right).$$

Note that $\lim_{j\to\infty} \tilde{h}_j(z) = h(p)$. Recall that we already have

$$0 \leq \varphi \leq h$$

in the proof of (A). This yields

$$0 \leq \tilde{\varphi}(z) \leq h(p). \tag{5.3}$$

Now, we consider two cases. When $p \in \partial X$, $h(p) = 0$ and hence $\tilde{\varphi} \equiv 0$. However, this contradicts to (5.2). Thus, the $\mathcal{C}^1$-estimate is proved in this case. On the other hand, when $p \in X$, since

$$dd^c\tilde{\varphi}_j \geq \frac{1}{C_j^2} dd^c\varphi_j \geq -\frac{1}{C_j^2}\omega,$$

we can see that $\tilde{\varphi}$ is psh on $\mathbb{C}^n$. However, $\tilde{\varphi}$ is uniformly bounded. Then, this yields that $\tilde{\varphi}$ is constant, but it contradicts to (5.2) again.

6. Proof of Main Theorem: $\mathcal{C}^2$-estimate

Finally, we shall treat the $\mathcal{C}^2$-estimate (E). We introduce the proof of Chu–Tosatti–Weinkove [CTW17]. Let $\lambda_1(\nabla^2\varphi) := \sup_{|V|=1} \nabla^2\varphi(V,V)$ be the maximum eigenvalue of $\nabla^2\varphi$. Observe that

$$\left|\nabla^2\varphi\right| \leq C\lambda_1(\nabla^2\varphi) + C.$$

This follows from

$$\left|\nabla^2\varphi\right| = \left(\sum_{j=1}^{2n} \lambda_j^2\right)^{\frac{1}{2}} \leq C\left(|\lambda_1| + |\lambda_{2n}|\right)$$

and

$$\sum_{j=1}^{2n} \lambda_j = \Delta_{\mathrm{d}}\varphi = 2\,\mathrm{tr}_\omega(\mathrm{dd}^c\varphi) \geq -2n.$$

Therefore, the main goal is to find a uniform upper bound for $\lambda_1(\nabla^2\varphi)$. To reach this aim, we want to apply maximum principle to

$$Q = \log\lambda_1(\nabla^2\varphi) + h(|\partial\varphi|^2_\omega) - A\varphi$$

where $h(s) = -\frac{1}{2}\left(1 + \sup_X |\partial\varphi|^2_\omega - s\right)$ and A is a constant to be determined. Note that Q is continuous on the domain $\{\lambda_1(\nabla^2\varphi) > 0\}$ and achieves a maximum at a point $x_0 \in X$ with $\lambda_1(\nabla^2\varphi(x_0)) > 0$. We may assume that x_0 is not on ∂X. Otherwise, we are done.

Unfortunately, Q may not be smooth, since the eigenspace associated to λ_1 may have dimension strictly greater than 1. Hence, we have to use some perturbation argument. Without loss of generality, we can say $(\varphi_{j\bar{k}})_{1\leq j,k\leq n}$ is diagonal at x_0 and $\varphi_{1\bar{1}} \geq \varphi_{2\bar{2}} \geq \cdots \geq \varphi_{n\bar{n}}$ and this implies $\tilde{g}_{1\bar{1}} \geq \cdots \geq \tilde{g}_{n\bar{n}}$. Let V_1 be a unit vector with respect to ω such that $\nabla^2\varphi(V_1, V_1) = \lambda_1$ at x_0. Let $\{V_1, V_2, \cdots, V_{2n}\}$ form an orthonormal basis of eigenvector of $\nabla^2\varphi$ at x_0 and we say $\lambda_1 \geq \cdots \geq \lambda_n$. We write $V_\beta = (V_\beta^1, \cdots, V_\beta^{2n}) = V_\beta^\alpha \frac{\partial}{\partial x^\alpha}$ which are components of V_β. Then, extend $V_1, \cdots, V_{2n}$ to be local vector fields in a neighborhood of x_0 by taking the components to be constants. Define a matrix $B = B_{\alpha\beta}\mathrm{d}x^\alpha \otimes \mathrm{d}x^\beta \in \Gamma(U_{x_0}, (T^*X)^{\otimes 2})$ near x_0 where

$$B_{\alpha\beta} := (\delta_{\alpha\beta} - V_1^\alpha V_1^\beta).$$

Note that $B(V_1, V_1 = 0)$ and $B(V_\mu, V_\mu) = 1$ for $\mu \neq 1$. Then, we define another tensor $\Psi \in \Gamma(U_{x_0}, \mathrm{End}(TX))$ by

$$\Psi^\alpha_\beta = g^{\alpha\gamma}\nabla^2_{\gamma\beta}\varphi - g^{\alpha\gamma}B_{\gamma\beta}.$$

Obviously, we have

$$\Psi(V_\mu) = (\Psi^\alpha_\beta V_\mu^\beta)_{1\leq\alpha\leq 2n} = (\lambda_\mu - 1)V_\mu + \langle V_1, V_\mu\rangle V_1,$$

and hence we obtain $\lambda_1(\Psi) = \lambda_1(\nabla^2\varphi)$ and $\lambda_\mu(\Psi) = \lambda_\mu(\nabla^2\varphi) - 1$ for $\mu \neq 1$ at x_0. Therefore, we can consider another quantity

$$\hat{Q} := \log \lambda_1(\Psi) + h(|\partial\varphi|^2_\omega) - A\varphi$$

which attains maximum at $x_0 \notin \partial X$. We want to apply the maximum principle on $\hat{Q}$, so we have to compute $\Delta_{\tilde{\omega}}\hat{Q}$. The following two lemmas are the key to finish the proof of the second order estimate, but the computations are too technical to be given here. One can see the original reference [CTW17] for more details.

Lemma 6.1 **([CTW17, Lemma 2.1]).** — *At x_0, we have*

$$\begin{aligned}
0 \geq \Delta_{\tilde{\omega}}\hat{Q} \geq & 2\sum_{\alpha>1} \frac{\tilde{g}^{i\bar{i}}\,|\partial_i(\varphi_{V_\alpha V_1})|^2}{\lambda_1(\lambda_1-\lambda_\alpha)} + \frac{\tilde{g}^{p\bar{p}}\tilde{g}^{q\bar{q}}\,|V_1(\tilde{g}_{p\bar{q}})|^2}{\lambda_1} - \frac{\tilde{g}^{i\bar{i}}\,|\partial_i(\varphi_{V_1V_1})|^2}{\lambda_1^2} \\
& + h'\sum_k \tilde{g}^{i\bar{i}}(|\varphi_{ik}|^2 + |\varphi_{i\bar{k}}|^2) + h''\tilde{g}^{i\bar{i}}\left|\partial_i\,|\partial\varphi|^2_\omega\right|^2 \qquad (6.1)\\
& + (A-C)\sum_i \tilde{g}^{i\bar{i}} - An,
\end{aligned}$$

where $\varphi_{\alpha\beta} = \nabla^2_{\alpha\beta}\varphi$, $\varphi_{V_\alpha V_\beta} = \varphi_{\gamma\delta}V_\alpha^\gamma V_\beta^\delta = \nabla^2\varphi(V_\alpha, V_\beta)$, and C is a constant depending only on background data and A_1.

Lemma 6.2 **([CTW17, Lemma 2.2]).** — *There is a uniform constant $C \geq 1$ which depends only on A_1 and background data such that if $0 < \varepsilon < \frac{1}{2}$ and $\lambda_1(x) \geq \frac{C}{\varepsilon^2}$, then at x_0 we have*

$$\begin{aligned}
\sum_i \frac{\tilde{g}^{i\bar{i}}\,|\partial_i(\varphi_{V_1V_1})|^2}{\lambda_1^2} \leq & 2(h')^2\tilde{g}^{i\bar{i}}\left|\partial_i\,|\partial\varphi|^2_\omega\right|^2 + 4\varepsilon A^2\tilde{g}^{i\bar{i}}\,|\varphi_i|^2 \\
& + 2\sum_{\alpha>1}\frac{\tilde{g}^{i\bar{i}}\,|\partial_i(\varphi_{V_\alpha V_1})|^2}{\lambda_1(\lambda_1-\lambda_\alpha)} + \frac{\tilde{g}^{p\bar{p}}\tilde{g}^{q\bar{q}}\,|V_1(\tilde{g}_{p\bar{q}})|}{\lambda_1} + \sum_i \tilde{g}^{i\bar{i}}. \qquad (6.2)
\end{aligned}$$

By Lemma 6.1 and Lemma 6.2, we have

$$\begin{aligned}
0 \geq & 2\sum_\alpha \frac{\tilde{g}^{i\bar{i}}\,|\partial_i(\varphi_{V_\alpha V_1})|^2}{\lambda_1(\lambda_1-\lambda_\alpha)} + \frac{\tilde{g}^{p\bar{p}}\tilde{g}^{q\bar{q}}\,|V_1(\tilde{g}_{p\bar{q}})|^2}{\lambda_1} - 2(h')^2\tilde{g}^{i\bar{i}}\left|\partial_i\,|\partial\varphi|^2_\omega\right|^2 \\
& - 4\varepsilon A^2\tilde{g}^{i\bar{i}}\,|\varphi_i|^2 - 2\sum_{\alpha>1}\frac{\tilde{g}^{i\bar{i}}\,|\partial_i(\varphi_{V_\alpha V_1})|^2}{\lambda_1(\lambda_1-\lambda_\alpha)} - \frac{\tilde{g}^{p\bar{p}}\tilde{g}^{q\bar{q}}\,|V_1(\tilde{g}_{p\bar{q}})|^2}{\lambda_1} - \sum_i \tilde{g}^{i\bar{i}} \qquad (6.3)\\
& + h'\sum_k \tilde{g}^{i\bar{i}}(|\varphi_{ik}|^2 + |\varphi_i\bar{k}|^2) + h''\tilde{g}^{i\bar{i}}\left|\partial_i\,|\partial\varphi|^2_\omega\right|^2 + (A-C)\sum_i \tilde{g}^{i\bar{i}} - An \\
= & -4\varepsilon A^2\tilde{g}^{i\bar{i}}\,|\varphi_i|^2 + h'\sum_k \tilde{g}^{i\bar{i}}(|\varphi_{ik}|^2 + |\varphi_{i\bar{k}}|^2) + (A-C_0)\sum_i \tilde{g}^{i\bar{i}} - An
\end{aligned}$$

if $0 < \varepsilon < \frac{1}{2}$ and $\lambda_1(x_0) \geq \frac{C}{\varepsilon^2}$. Choose $A = C_0 + 2$ and $\varepsilon = \frac{1}{4A^2(\sup_X|\partial\varphi|^2_\omega + 1)}$. At x_0, we get

$$0 \geq -\sum_i \tilde{g}^{i\bar{i}} + h'\sum_k \tilde{g}^{i\bar{i}}(|\varphi_{ik}|^2 + |\varphi_{i\bar{k}}|^2) + 2\sum_i \tilde{g}^{i\bar{i}} - An$$

and this implies

$$\sum_i \tilde{g}^{i\bar{i}} + h' \sum_k \tilde{g}^{i\bar{i}}(|\varphi_{ik}|^2 + |\varphi_{i\bar{k}}|^2) \leq C.$$

Note that $h' > \frac{1}{2+2\sup_X|\nabla\varphi|} > 0$ by assumption and $\sum_i \tilde{g}^{i\bar{i}} \geq 1 > 0$ because $\sum_i \tilde{g}^{i\bar{i}} \geq n\sqrt[n]{\prod_i g^{i\bar{i}}} \geq ne^{\frac{-F}{n}} \geq n$. Hence, we can say $|\varphi_{ik}|^2, |\varphi_{i\bar{k}}|^2$ are uniformly bounded at x_0 and thus $\lambda(x_0)$ is uniform ly bounded. This completes the proof of (E) as well as the proof of Theorem 3.1.

7. Concluding remarks

7.1. No $\mathcal{C}^2$-solution in general. — In this section, we will give a brief overview of the work of Darvas-Lempert-Vivas [**DL12**, **LV13**, **Dar14**] who proved there is no $\mathcal{C}^2$-solution in general.

First, we consider the Dirichlet problem with the following boundary condition:

$$\begin{cases} (\omega + dd^c\Phi)^{n+1} = 0 \\ \Phi(x,t,s) = \Phi(x,t) \text{ for } (x, e^{t+is}) \in X \times A \\ \Phi(x,t) = \begin{cases} 0 & \text{when } t = 0 \\ v(x) & \text{when } t = 1. \end{cases} \end{cases} \tag{7.1}$$

We always consider here that X is a compact complex manifold endowed with a holomorphic isometry f with a isolated fixed point x_0 (e.g. X is a complex torus $\mathbb{C}^n/\Gamma$ and f is the map induced by $\tilde{f}: \mathbb{C}^n \to \mathbb{C}^n$ where $\tilde{f}: z \mapsto -z$).

***Theorem 7.1* ([DL12]). —** *Suppose a compact Kähler manifold (X,ω) admits a holomorphic isometry $f: X \to X$ with a isolated fixed point x_0, and $f^2 = \mathrm{Id}_X$. Then, there is a function $v \in \mathcal{K}_\omega$ for which (7.1) has NO ω-psh solution $\Phi \in \mathcal{C}^{\partial\bar{\partial}}(X \times A)$ where*

$$\mathcal{C}^{\partial\bar{\partial}}(X) := \{w \in \mathcal{C}(X) \,|\, \text{the current } dd^c(w) \text{ is represented by a form continuous on } \bar{X}\}.$$

Note that $\mathcal{C}^2(X) \subset \mathcal{C}^{\partial\bar{\partial}}(X)$. One can choose v to satisfies $f^*v = v$.

The strategy of the proof is the following: First, by the Perron method point of view, we know the solution $\Phi = \sup_{v\in V} v$ where V is the envelope of all subsolutions of (7.1). The envelope V is invariant under f and hence Φ is invariant, namely $\Phi(x,t) = \Phi(f(x),t)$. Secondly, we notice that $\Phi(x_0,t)$ is harmonic with respect to the coordinate (t,s) and since Φ does not depend on s, we then have $\Phi(x_0,t) = a \cdot t$. Finally, using the Poisson integral formula, one can obtain an estimate on the boundary function v

$$\left|\sum_{j,k=1}^n v_{jk}(x_0)\xi_j\xi_k\right| \leq \sum_{j,k=1}^n (2g_{j\bar{k}}(x_0) + v_{j\bar{k}}(x_0))\xi_j\bar{\xi}_k \tag{7.2}$$

for $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{C}^n$ and this estimate is sharp. Conversely, $(v_{j\bar{k}}(x_0)) = (p_{j\bar{k}})$ and $(v_{jk}(x_0)) = (q_{jk})$ can be arbitrarily prescribed for f-invariant $v \in \mathcal{K}_\omega$ as long as $(g_{j\bar{k}}(x_0) + p_{j\bar{k}})$ is positive-definite. These complete the proof of Theorem 7.1

7.2. Toric setting. — Although the theorem of Darvars-Lempert-Vivas tells us there is no smooth geodesic in general, Daniel Guan showed that if we only consider the toric cases it is possible to have a smooth geodesic.

***Theorem 7.2* ([Gua99]). —** *Suppose (X, ω) is a compact toric manifold and ω_0, ω_1 are two toric metrics in $[\omega]$. Then the Mabuchi geodesic connecting ω_0 and ω_1 is smooth.*

A toric potential corresponds to a convex function $F : \mathbb{R}^n \to \mathbb{R}$ via the log map. The Mabuchi geodesic equation for φ_t corresponds to a similar equation for F_t. Applying Legendre transformation to F_t, we obtain an equation for G_t which turns out to be $\partial_t^2 G_t = 0$, hence $t \mapsto G_t$ is affine and F_t is smooth.

7.3. Singular setting. — One can also consider these problems when X is a mildly singular variety. In 2019, Chu-McCleerey [CM19] proved a similar version of the $\mathcal{C}^{1,1}$-estimate in this singular context.

***Theorem 7.3* ([CM19]). —** *Given two cohomologous Kähler metrics ω_1, ω_2 on a singular Kähler variety X, the geodesic connecting them is in $\mathcal{C}^{1,1}_{\text{loc}}(X_{\text{Reg}} \times A)$, where $A \subset \mathbb{C}$ is an annulus and X_{Reg} is the smooth part of X.*

The idea of the proof is first using Hironaka's theorem to resolve the singularities $p : \tilde{X} \to X$. The Kähler metric ω on X can be pulled-back to the resolution space $\tilde{X}$. An important difficulty is that $p^*\omega$ is no longer Kähler on $\tilde{X}$.

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